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Perturbative Casimir energies of dispersive spheres, cubes and cylinders

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Abstract

The *quantum-electrodynamic binding energies* B are determined perturbatively to order $(n\alpha)^2$ for single macroscopic bodies (quasi-continua mimicking atomic solids) having the dispersive dielectric function $\varepsilon(\omega) \simeq \{1 + 4\pi n\alpha\Omega^2/(\Omega^2 - (\omega^2 - \mathrm{i}0)^2)\}$, as if each atom were an oscillator of frequency Ω, and n the number density of atoms (pairwise separations ρ). The familiar divergences all persist although they are modified by dispersion (finite rather than infinite Ω); they must be controlled instead by imposing the condition $\rho > \lambda \sim$(minimum lattice spacing) $\ll c/\Omega$. QED gives identically the same $B = -(n\alpha)^2(1/2)\int_\lambda^\infty \mathrm{d}\rho\, \rho^2 f(\rho)g(\rho)$ as one obtains from the properly retarded attraction $-\alpha^2 f(\rho)$ between atoms, with $g(\rho)$ a correlation function defined purely by the geometry of the body. The first three terms of the Taylor series for g are determined, respectively, by volume V, surface area S and any sharp edges. To order $(n\alpha)^2$, but not beyond, the results for solid bodies lead directly to those for cavities of the same shape and size in otherwise unbounded material.

Unlike the attraction between disjoint bodies, B for any single finite body (typical linear dimensions $a \gg c/\Omega$) is dominated by components proportional, respectively, to $(n\alpha)^2\hbar\Omega \times \{-V/\lambda^3, +S/\lambda^2, -a/\lambda$ (if there are edges) and $\pm\log(c/2\Omega\lambda)\}$. These always tend to induce collapse rather than expansion. The *pure Casimir components* are of order $(n\alpha)^2\hbar c/a$, and (like the logarithmic terms) sometimes positive, which makes them impossible to understand if the dominant terms are disregarded. The B are found in closed form for spheres, spherical shells and cubes, up to corrections vanishing with λ. For unit length of an infinitely long right circular cylinder of radius a, the standard V- and S-proportional terms are corrected only by $-(n\alpha)^2(\pi^2\hbar\Omega/128a)\log(c/2\Omega\lambda)$; the pure Casimir component, which would be proportional to $(n\alpha)^2\hbar c/a^2$, vanishes through apparently accidental cancellations peculiar to order $(n\alpha)^2$.

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1. Introduction

Casimir energies in a general sense stem from the quantum-mechanical interaction of the electromagnetic field with macroscopic bodies modelled as continua. More narrowly, we shall describe as *pure Casimir terms* those components of such energies that (a) depend only on the electro*static* polarizability of the material, and (b) for finite bodies are proportional[1] to $\hbar c/a$, where a is some length characterizing the geometry of the body, for example the radius of a sphere or the edge-length of a cube. (For an indefinitely extended cylinder, the pure Casimir terms are those proportional to $\hbar c/a^2$ per unit length.) Such effects are often thought glamorous because they are ascribed to the total zero-point (ground-state) energy $\sum_i \omega_i/2$ of the system, where the ω_i are the frequencies of the classical normal modes, affected by the bodies through the matching conditions on the field at their surfaces. In practice $\sum_i$ is shorthand for an integral or a sum of integrals, which always diverge. Traditionally this is remedied by introducing a cutoff (a convergence factor), for example $\exp(-\tilde{\lambda}\omega)$, which suppresses the contribution of high frequencies; calculating whatever observable one requires; and taking the no-cutoff limit $\tilde{\lambda} \to 0$, $\exp(-\tilde{\lambda}\omega) \to 1$ at the end. Those components of the ground-state energy that would become infinite in this limit we call (nominally) divergent; those that remain finite we call convergent, and those that vanish in the limit we shall drop without further comment. In all the examples we study it turns out that the convergent components are precisely the pure Casimir terms.

The justification usually advanced for this procedure cites the fact that real materials, being dispersive, become transparent at high frequencies, and then claims that, in consequence, modes with $\omega > 1/\tilde{\lambda}$ remain essentially unaffected by the bodies, whence their contributions could have been ignored in the first place. We shall see that as regards the self-energy of any single connected body (as opposed to the potential between mutually disjoint bodies) this argument is wrong on both counts: transparency at high frequencies does not eliminate divergences; and the physics is dominated by the divergent components of the energy, the pure Casimir terms being so much smaller that they will never become observable.

Indeed, insofar as the classic papers are devoted largely to perfect reflectors, they can cause confusion in three ways. (i) In the original problem of two parallel mirrors, they manage to get the correct attractive energy by sidestepping divergent terms altogether[2], fostering a presumption that divergences should be irrelevant quite generally, whereas for connected bodies they are not. (ii) Thus encouraged, they eliminate divergences from the energy of the spherical shell (perhaps the simplest of connected bodies), producing a finite pure Casimir energy which taken by itself happens to be repulsive. This is commonly perceived as strange, in that total enclosure of a finite region inside the shell appears to raise the energy, whereas partial enclosure of an infinite region between parallel mirrors had appeared to lower it[3]. (iii) They are liable to mislead simply through having started directly in the limit of perfect reflection, whereas the physics becomes visible only as this limit is approached.

[1] We shall use natural units $\hbar = 1 = c$, and rationalized Gaussian units for the Maxwell field. Polarizability is defined so that in an atom an electric field E induces a dipole moment αE and a Stark shift $-\alpha E^2/2$, while in a volume element $\mathrm{d}V$ it induces a moment $E\,\mathrm{d}V\,(\varepsilon - 1)/4\pi$. These dispositions of 4π preserve the traditional form $-23\alpha^2/4\pi\rho^7$ of the long-range interatomic potential.

[2] This is the approach extended to disjoint imperfect reflectors in the so-called Lifshitz theory, lucidly discussed by Milonni (1994).

[3] The pure Casimir energy for the shell was found by Boyer (1968). It was confirmed by Davies (1972), the first to note that the positive sign was paradoxical not only by contrast with parallel mirrors, but because it should apply equally to shells made of ordinary materials, whose molecules attract rather than repel each other. Though echoes of his observation are discernible in the papers of Candelas (1982) and of Ambjorn and Wolfram (1983), it was, unfortunately, not followed through at the time. For further references on spheres see section 7.

On the other hand, more recent work has repeatedly addressed $\sum_i \omega_i/2$ for imperfectly reflecting dielectric spheres, deploying considerable sophistication with Bessel functions in order to master the normal modes, and also with (mainly dimensional) regularization methods in order to eliminate divergences. For relative simplicity, many such calculations take the material as optically dilute ($\varepsilon - 1 \ll 1$) and also as nondispersive (ε independent of ω). But under these conditions it is far easier to obtain the ground-state energy as a perturbative shift relative to the true vacuum (Maxwell field in empty, body-absent space), with the coupling between body and field treated as the perturbation. That is what we shall do, exploiting the further advantage of perturbation theory, that it can deal with dispersive materials just as easily as with nondispersive ones. Furthermore, perturbation theory is so tightly constrained as to be almost foolproof, with answers explicit to a point where the underlying physics, especially the status of cutoffs, becomes unmistakeably obvious.

Accordingly, we adopt a model designed to mimic optically dilute atomic solids, *working only to order* $(n\alpha)^2 \ll 1$, with n the number density of atoms and α the electrostatic atomic polarizability. Each atom is treated as a simple harmonic oscillator, so that its dynamic polarizability reads

$$\alpha(\omega) = \alpha\Omega^2/\big(\Omega^2 - (\omega^2 - \mathrm{i}0)^2\big) \tag{1.1}$$

with Ω the renormalized frequency. The dielectric function

$$\varepsilon(\omega) \simeq 1 + 4\pi n\alpha(\omega)$$

is perfectly causal (i.e. compatible with the Kramers–Kronig relation), becoming nondispersive in the limit $\Omega \to \infty$. On the other hand, acccording to the Clausius–Mossotti formula $\varepsilon = (1 + 8\pi n\alpha/3)/(1 - 4\pi n\alpha/3)$, perfect reflection at $\omega = 0$ (an infinite dielectric constant) ensues for $n\alpha = 3/4\pi$, obviously out of reach under our assumptions[4].

Applied to this model, quantum-electrodynamic perturbation theory shows that the energy shift $\Delta E^{(1)}$ of order $n\alpha$ (formally of order e^2) is just the sum of the individual atomic Lamb shifts, which we disregard (see e.g. Milonni *et al* 1999 and references there). What we require is the shift $\Delta E^{(2)}$ of order $(n\alpha)^2$ (formally of order e^4): eventually, $\Delta E^{(2)}$ turns out to be identically the same as the binding energy B of the body calculated from pairwise ('additive') potential energies $\mathrm{d}^3r\,\mathrm{d}^3r'\,U(\rho)$ between any two volume elements at $\boldsymbol{r}, \boldsymbol{r}'$. Here

$$\boldsymbol{\rho} \equiv \boldsymbol{r} - \boldsymbol{r}', \qquad U(\rho) = -(n\alpha)^2 f(\rho), \qquad [f] = [L]^{-7} \tag{1.2}$$

and $\alpha^2 f(\rho)$ is the standard dipole approximation to the interatomic potential; f modulates from $f_{\mathrm{vdW}} \equiv 3\Omega/4\rho^6$ in the nonretarded Van der Waals (VdW) regime, where $\rho\Omega \ll 1$, to $f_{\mathrm{CP}} \equiv 23/4\pi\rho^7$ in the fully retarded Casimir–Polder (CP) regime, where $\rho\Omega \gg 1$, so that formally this regime can be stretched to all ρ by taking the prior nondispersive limit $\Omega \to \infty$.

It is these binding energies that we calculate. Clarity and efficiency depend on writing each B as proportional to a convolution $\int_0^\infty \mathrm{d}\rho\,\rho^2 f(\rho)g(\rho)$ of $f(\rho)$ with a two-point correlation function $g(\rho)$ determined purely by the geometry of the body. Remarkably, the first few coefficients in the Taylor series for $g(\rho)$ are dictated by the volume V, the total surface area S and the local geometry of the surface, in particular by the principal radii of curvature and the opening angles Φ of any sharp edges.

Given additive potentials, i.e. accurately to $\mathcal{O}(n\alpha)^2$ but not beyond (Milonni and Lerner 1992), our task is simplified by the *cavity theorem*, which expresses the energy of a cavity in terms of the energy of a body having the same shape and size.

[4] Nor can we address media having refractive index $\sqrt{\varepsilon\mu} = 1$ at all frequencies, because we cannot afford to treat dispersion too unrealistically, while models yielding such μ would imply magnetic contributions to U quite inconceivable for real atoms (see e.g. Au 1972, Power 1974, Salam 2000, Farina *et al* 2000). Lastly, we consider only zero temperature, and only three-dimensional space.

Subject to the cutoff envisaged above, perturbation theory yields $\Delta E^{(2)}$, initially, in the form $-\iint \mathrm{d}^3k_1\,\mathrm{d}^3k_2\,\exp\left[-\tilde{\lambda}(k_1+k_2)\right]\ldots$, with the dots representing squared matrix elements and energy denominators. This is the form originally considered by the writer (Barton 1999, to be referred to as I) while trying to deal with nondispersive materials. It leads to a potential $f(\rho,\tilde{\lambda})$ which correctly reproduces f_{CP} at large ρ, but tends to a finite limit as $\rho\to 0$. Such behaviour might be deemed to tally with the traditional role of $1/\tilde{\lambda}$ as a mere placeholder for some transparency parameter like Ω; but for the small values of ρ that dominate B it is far too unrealistic to be trusted. One can see this by applying the same cutoff prescription to the potential between point charges rather than atoms: the Coulomb potential would be modified, not Yukawa-like at large distances in some manner interpretable through screening, but by replacing its $1/\rho$ singularity as $\rho\to 0$ with a constant limit of order $1/\tilde{\lambda}$. On the other hand, without the cutoff the convolution for B makes no sense, because $f(\rho\to 0)\sim 1/\rho^6$ would cause it to diverge at its lower limit.

Fortunately, the solution to the problem is obvious from the context: abandon fictitious frequency cutoffs like $\exp(-\tilde{\lambda}\omega)$, revert to the true $f(\rho)$, and remedy the divergence in B by adopting a purely geometric restriction $\rho>\lambda$, where λ is the minimum distance between atoms, comparable to the radius of the repulsive core of the interatomic potential. One need merely rewrite the convolution as $\int_\lambda^\infty \mathrm{d}\rho\,\rho^2 f(\rho)g(\rho)$, convergence being secured not by a vain appeal to dispersion, but by this minimal recognition that real materials are atomically granular rather than truly continuous. In particular, the new cutoff length λ is far shorter than the traditional $\tilde{\lambda}$, roughly in the ratio of atomic radius to typical excitation wavelength. Nominal divergence is redefined as divergence in the hypothetical limit $\lambda\to 0$.

The energies B of *finite bodies* emerge, to order $(n\alpha)^2$, in the form

$$Vu + S\sigma + \int \mathrm{d}L\,\zeta(\Phi(L)) + (\text{s–d}) + (\text{pure Casimir})$$

where $u<0$ is the binding-energy density in unbounded material; $\sigma>0$ is the surface tension; $\int \mathrm{d}L\,\zeta<0$ runs over any sharp edges and the *shape-dependent term* (s–d) depends *only* on the shape and not on the size of the body. All but the pure Casimir terms diverge, with $u\sim\Omega/\lambda^3$, $\sigma\sim\Omega/\lambda^2$, $\zeta\sim\Omega/\lambda$ and (s–d) $\sim\Omega\log(1/2\Omega\lambda)$ respectively. The nondispersive limit taken at the outset (entailing $f\to f_{\mathrm{CP}}$ for all ρ) would aggravate these divergences to $1/\lambda^4$, $1/\lambda^3$, $1/\lambda^2$ and $1/\lambda$ respectively. However, it makes no sense to try and identify separate retarded and VdW contributions to f and thereby to B. On the formal level, f changes perfectly smoothly between the two regimes. More to the point, only in the Coulomb gauge can one ascribe the familiar nonretarded potential f_{VdW} wholly to the unquantized longitudinal field, with retardation corrections calculated separately as effects of the quantized transverse field; whereas after a gauge change from minimal to '$\boldsymbol{d}\cdot\boldsymbol{E}$' coupling, the interaction with the quantized transverse field delivers the correct $f(\rho)$ for all ρ through one and the same calculation.

Since B is dominated by $Vu+S\sigma$, it always tends to induce contraction rather than expansion; from this point of view the signs of the pure Casimir terms are irrelevant, simply because these terms are so much smaller. On the other hand, Vu and $S\sigma$ are standard features of condensed-state physics, and one might reasonably choose to adopt a renormalization procedure which absorbs our calculated u and σ into the measured values of these quantities. This would leave one free to concentrate, if one wished, on the other components of B; but then one must remember that the physically dominant energies have been removed from view, and not read paradoxes into the signs of those that are left visible. Thus, *renormalization in the (only) sense appropriate to Casimir effects is not in any way analogous to renormalization in conventional quantum field theory*. Regarding the latter, one knows from the outset that the divergences one encounters are spurious, and the only problem is to eliminate them as quickly

as possible. Regarding the former, it may be convenient to organize a calculation so that the nominal divergences are standardized and isolated; but they must be properly evaluated and borne in mind, because they embody essentials of the physics.

To substantiate this summary, we shall determine the Casimir energies of spheres and spherical shells, cubes and right circular cylinders (radius a, infinite length L). Spheres are the simplest connected shapes. The results for cubes serve to discourage premature generalization from spheres, and feature the simplest-to-calculate contributions from sharp edges. Cylinders furnish a classic example of the failure of plausible anticipation: one finds $[B - (Vu + S\sigma)]/L \sim -(n\alpha)^2(\Omega/a)\log(1/2\Omega\lambda)$, divergent yet proportional to $1/a$, and with no pure Casimir term at all.

The rest of this paper is laid out as follows.

Section 2 spells out the potential (1.2) and the provenance of the cutoff λ; and quotes (from appendix A) the moments $\mathcal{J}_N(\lambda) \equiv \int_\lambda^\infty \mathrm{d}\rho\, f(\rho)\rho^N$, which prove central because all nominal divergences enter through the $\mathcal{J}_N(\lambda)$ with N from 1 to 5. Just how, to order $(n\alpha)^2$, the potential does duty for quantum field theory is explained, somewhat pedantically, in appendix E; while appendix F cites measured properties of solid argon to show that working only to this order can be quite reasonable.

Section 3 proves the cavity theorem. Appendix B demonstrates the equally important fact that the theorem fails for non-additive (eg three-body) potentials, whose contributions can matter to order $(n\alpha)^3$ and beyond: for instance, the Casimir energies of strongly reflecting solid spheres fail to reveal the Casimir energies of spherical cavities cut into the same material.

Section 4 explains the renormalization procedure for the total energy B. The two strongest divergences, proportional to $\mathcal{J}_2(\lambda)$ and $\mathcal{J}_3(\lambda)$, are absorbed by redefinitions of u and σ respectively; others are assigned to the renormalized energy ΔB. Only for parallel-plane geometries is ΔB wholly convergent; for spheres and cubes it retains shape-dependent divergences proportional to $\mathcal{J}_5(\lambda)$; sharp edges produce stronger divergences proportional to $\mathcal{J}_4(\lambda)$. The crucially important two-point correlation function $g(\rho)$ is discussed in section 5. The first few terms of its Taylor series determine what divergences occur, and govern the role of the $\mathcal{J}_N(\lambda)$. These terms are derived in appendix C, with the edge-dependent function $\zeta(\Phi)$ by far the most awkward.

The tools thus provided are used in sections 6–10 to determine ΔB for gaps between half-spaces, spheres, spherical shells and cubes. (Section 9 is a brief digression on an infinitesimally thin but finite-mass spherical shell, calculating the forces directly, instead of leaving them implicit in the expression for B, as we leave them everywhere else.) Section 11 considers the right circular cylinder, with remarkable results we have already mentioned. Unlike spheres and cubes, cylinders are not finite bodies; this makes the technicalities quite challenging, and they have been relegated to appendix D.

With our main conclusions already outlined in this section, and verified in the sequence just laid out, the final section 12 re-states them only briefly and dogmatically; then it closes with some general comments, including one on deceptive folklore about perfect reflectors.

2. The potential

In our continuum model, the mutual potential energy of two volume elements is given by (1.2), with

$$f(\rho) = \frac{\Omega^4}{\pi\rho^3}\int_0^\infty \mathrm{d}x\, \exp(-2x)\frac{h(x)}{\left[x^2 + (\Omega\rho)^2\right]^2}, \qquad h(x) = 3 + 6x + 5x^2 + 2x^3 + x^4. \quad (2.1)$$

The function $f(\rho)$ is derived for example by Feinberg *et al* (1989) and by Power and Thirunamachandran (1985, 1994). It can be expressed in terms of $\mathrm{Ci}(M) \equiv -\int_M^\infty \mathrm{d}t\ \cos(t)/t$ and $\mathrm{si}(M) \equiv -\int_M^\infty \mathrm{d}t\ \sin(t)/t$ as

$$f = -\frac{32\Omega^7}{\pi M^4}\frac{\partial}{\partial M}\left\{\frac{1}{M}h_{\mathrm{op}}\left[\sin M\mathrm{Ci}(M) - \cos M\mathrm{si}(M)\right]\right\}, \qquad M \equiv 2\Omega\rho \tag{2.2}$$

$$h_{\mathrm{op}} \equiv 3 - 3M\frac{\partial}{\partial M} + \frac{5}{4}M^2\left(\frac{\partial}{\partial M}\right)^2 - \frac{1}{4}M^3\left(\frac{\partial}{\partial M}\right)^3 + \frac{1}{16}M^4\left(\frac{\partial}{\partial M}\right)^4. \tag{2.3}$$

As explained in section 1, we impose a cutoff

$$f(\rho < \lambda) = 0, \qquad \Omega\lambda \ll 1.$$

This, though a fiction, is a convenient shortcut to somewhere near the truth. At small separations, overlap between the electron clouds makes the interatomic potential highly repulsive, i.e. large and positive rather than zero. In the limit of an infinitely hard core, atoms cannot approach each other closer than the core radius λ. Hence, for real materials (as opposed to continuum models) the correlation function $g(\rho)$, to be introduced presently, should vanish for $\rho < \lambda$; and so should the product gf, the only combination through which f ever enters. The consequences are the same as if f itself were cut off, in the way just written. The product $\Omega\lambda$ is small because λ is of the order of the interatomic spacing, while $2\pi/\Omega$ is a typical emission wavelength: for solid argon appendix F estimates $\Omega\lambda \sim 0.014$.

Since we rely on a hard-core potential, we cannot sensibly consider variations of density, and must treat n as fixed. Thus, given a fixed amount of material, the total volume V is also fixed. Moreover, by disregarding crystal structure the model disregards rigidity. The upshot is that the energies we shall calculate cannot easily yield the local stresses for which they must ultimately be responsible.

In the VdW regime $\Omega\rho \ll 1$ one has

$$f = \frac{\Omega}{\rho^6}\left\{\frac{3}{4} - \frac{1}{4}(\Omega\rho)^2 + \frac{7}{6\pi}(\Omega\rho)^3 - \frac{3}{4}(\Omega\rho)^4 + \frac{1}{15\pi}\left[44\log\left(\frac{1}{2\Omega\rho}\right) - 44\gamma + \frac{607}{15}\right](\Omega\rho)^5 + \cdots\right\} \tag{2.4}$$

where $\gamma =$ Euler's constant $\simeq 0.577$. The leading term is

$$f_{\mathrm{VdW}} \equiv C_{\mathrm{VdW}}\Omega/\rho^6, \qquad C_{\mathrm{VdW}} \equiv 3/4. \tag{2.5}$$

In the CP regime $\Omega\rho \gg 1$ one has

$$f = \frac{1}{\pi\rho^7}\left\{\frac{23}{4} - \frac{129}{4}\frac{1}{(\Omega\rho)^2} + \frac{1917}{4}\frac{1}{(\Omega\rho)^4} - \frac{24\,075}{2}\frac{1}{(\Omega\rho)^6} + \cdots\right\}. \tag{2.6}$$

Here we write the leading term as

$$f_{\mathrm{CP}} \equiv C_{\mathrm{CP}}/\rho^7, \qquad C_{\mathrm{CP}} \equiv \frac{1}{\pi}\int_0^\infty \mathrm{d}x\ \exp(-2x)h(x) = 23/4\pi; \tag{2.7}$$

it emerges from (2.1) on replacing $\left[x^2 + (\Omega\rho)^2\right]^2 \to (\Omega\rho)^4$.

The *nondispersive* version of our model has $\alpha(\omega) = \alpha$ at all frequencies however high, as if the restoring forces specified in appendix E.2 were arbitrarily strong. Formally it is obtainable by taking the limit $\Omega \to \infty$ right at the start, which replaces $f(\rho)$ by $f_{\mathrm{CP}}(\rho)$ at *all* values of ρ. We call this limit 'formal' because it misrepresents the physics quite significantly: for instance, it will be seen to turn logarithmic divergences into linear ($\Omega\log(1/\lambda)$ into $1/\lambda$), linear into quadratic and so on.

By contrast, we cannot implement the opposite limit $\Omega \to 0$ (no restoring forces), because the electromagnetic response of delocalized charge carriers (e.g. of the electron plasma in metals[5]) is so utterly different from that of insulators. Mathematically, the difference is reflected by the fact that at $\omega = 0$ the polarizability of a plasma has a pole, incompatibly with the implication of (1.1) that $\lim_{\Omega\to 0} n\alpha(\omega) = 0$.

Crucial to all our calculations are the moments

$$\mathcal{J}_N(x) \equiv \int_x^\infty \mathrm{d}\rho\, f(\rho)\rho^N, \qquad [\mathcal{J}_N] = [L]^{-6+N}. \tag{2.8}$$

Since $f(\rho \to \infty) \sim C_{\mathrm{CP}}/\rho^7$, the $\mathcal{J}_{N\geqslant 6}$ would diverge. If (wrongly) f_{VdW} were retained as $\rho \to \infty$, then there would be a divergence already in $\mathcal{J}_5$.

In the CP regime one evaluates the $\mathcal{J}_N$ with f from (2.6); the leading terms are

$$\mathcal{J}_N(x) \approx \left(\frac{C_{\mathrm{CP}}}{6-N}\right)\frac{1}{x^{6-N}}. \tag{2.9}$$

The VdW regime $\Omega x \ll 1$ includes $x = \lambda$, for which we quote from appendix A:

$$\mathcal{J}_1(\lambda)/\Omega^5 = \frac{3}{16}\left(\frac{1}{\Omega\lambda}\right)^4 - \frac{1}{8}\left(\frac{1}{\Omega\lambda}\right)^2 + \frac{7}{6\pi}\left(\frac{1}{\Omega\lambda}\right) + \left[-\frac{3}{4}\log\left(\frac{1}{2\Omega\lambda}\right) + \frac{3\gamma}{4} - \frac{1}{8}\right] + \mathcal{O}(\Omega\lambda) \tag{2.10}$$

$$\mathcal{J}_2(\lambda)/\Omega^4 = \frac{1}{4}\left(\frac{1}{\Omega\lambda}\right)^3 - \frac{1}{4}\left(\frac{1}{\Omega\lambda}\right) + \left[\frac{7}{6\pi}\log\left(\frac{1}{2\Omega\lambda}\right) - \frac{7\gamma}{6\pi} + \frac{35}{36\pi}\right] + \mathcal{O}(\Omega\lambda) \tag{2.11}$$

$$\mathcal{J}_3(\lambda)/\Omega^3 = \frac{3}{8}\left(\frac{1}{\Omega\lambda}\right)^2 + \left[-\frac{1}{4}\log\left(\frac{1}{2\Omega\lambda}\right) + \frac{\gamma}{4} - \frac{3}{16}\right] + \mathcal{O}(\Omega\lambda) \tag{2.12}$$

$$\mathcal{J}_4(\lambda)/\Omega^2 = \frac{3}{4}\left(\frac{1}{\Omega\lambda}\right) - \frac{11}{8\pi} + \mathcal{O}(\Omega\lambda) \tag{2.13}$$

$$\mathcal{J}_5(\lambda)/\Omega = \frac{3}{4}\log\left(\frac{1}{2\Omega\lambda}\right) - \frac{3\gamma}{4} + \frac{65}{32} + \mathcal{O}\big(\Omega\lambda\log(1/\Omega\lambda)\big). \tag{2.14}$$

For example, the energy density u in an unbounded medium reads

$$u \equiv \frac{1}{2}\int \mathrm{d}^3\rho\, U(\rho) = -(n\alpha)^2\, 2\pi\, \mathcal{J}_2(\lambda) \simeq -(n\alpha)^2\,\frac{\pi}{2}\,\frac{\Omega}{\lambda^3}. \tag{2.15}$$

Next, the surface tension σ (energy per unit surface area) can be identified from the special case of a halfspace, by adding up the potential deficiencies of volume elements at a depth z below the surface, as compared with elements at infinite depth. One finds

$$\sigma \equiv \frac{1}{2}(n\alpha)^2 2\pi \int_0^\infty \mathrm{d}z \int_z^\infty \mathrm{d}\rho\, \rho^2 f(\rho) \int_{z/\rho}^1 \mathrm{d}\cos\,\theta = (n\alpha)^2\,\frac{\pi}{2}\mathcal{J}_3(\lambda) \simeq (n\alpha)^2\,\frac{3\pi}{16}\,\frac{\Omega}{\lambda^2}. \tag{2.16}$$

A third quantity of a somewhat similar kind is the energy per unit length of a long sharp edge, given by

$$\zeta(\Phi) = -(n\alpha)^2\frac{1}{3}Z(\Phi)\mathcal{J}_4(\lambda) \simeq -(n\alpha)^2\,\frac{1}{4}Z(\Phi)\frac{\Omega}{\lambda}, \qquad Z(\Phi) \equiv 1 + \frac{(\pi-\Phi)}{\tan(\Phi)} \tag{2.17}$$

where Φ is the angle of the edge (see figure C.1): note the relation $\zeta(\Phi) = \zeta(2\pi - \Phi)$, which tallies with the cavity theorem proved in the next section. The correlation function $g(\rho)$

[5] A nonrelativistic model of metals is considered elsewhere, without recourse to perturbation theory (Barton 1979).

defined in section 5 leads to all these parameters systematically: to u and σ almost at once, with $\zeta(\Phi)$ deferred to appendix C.

Loosely speaking, the signs of u, σ, ζ alternate because σ corrects for fewer neighbouring volume elements near a surface, while ζ corrects for fewer neighbouring surface elements near an edge. Since σ is positive, it tends to minimize surface area. Since ζ is the same whether the wedge is convex or concave, and varies from $\zeta(0) = -\infty$ to $\zeta(\pi) = 0$, it tends to minimize Φ, i.e. to close wedges of either kind[6].

3. The cavity theorem

3.1. Definitions and statement

Define the characteristic function of a material body by

$$\chi(\boldsymbol{r}) = \begin{cases} 1 & \text{inside} \\ 0 & \text{outside.} \end{cases}$$

Binding energies are called B:

$$B_{\mathrm{b}} \equiv B(\text{body}) = \tfrac{1}{2} \iint \mathrm{d}^3 r \, \mathrm{d}^3 r' \, \chi(\boldsymbol{r}) \chi(\boldsymbol{r}') U(\rho). \tag{3.1}$$

Then, formally,

$$B(\text{unbounded medium}) \equiv B^* = (\text{total volume}) \times u \equiv V^* u$$

where V^* and therefore B^* are of course infinite.

We write the total energy of *a body* (suffix b as above) occupying a region with volume V and surface area S as

$$B_{\mathrm{b}} \equiv Vu + S\sigma + \Delta B_{\mathrm{b}}. \tag{3.2}$$

The corresponding system consisting of unbounded medium but with the region formerly occupied by the body now empty is called *a cavity* (suffix c). The energy B_{c} of this system is infinite, but we define an associated finite energy

$$B_{\mathrm{c}} - B^* \equiv -Vu + S\sigma + \Delta B_{\mathrm{c}}. \tag{3.3}$$

To appreciate the significance of ΔB_{c}, consider two systems, (i) a very large solid body, and (ii) the same body containing, deep inside, a much smaller cavity c of volume V and surface area S, plus a second solid body b removed arbitrarily far from the first, having the same shape and size as the cavity. Their energy difference is

$$E(ii) - E(i) \simeq 2S\sigma + \Delta B_{\mathrm{b}} + \Delta B_{\mathrm{c}}.$$

Theorem. *To order* $(n\alpha)^2$ *one has*

$$\Delta B_{\mathrm{c}} = \Delta B_{\mathrm{b}} \equiv \Delta B. \tag{3.4}$$

Appendix B shows that to order $(n\alpha)^3$ the theorem fails on account of three-body interactions. On the other hand, for strictly homogeneous media, i.e. *assuming strictly constant* n, the two-body interactions $U(\rho)$ respect the theorem *exactly*: for instance, violations to order $(n\alpha)^4$ can in that case stem only from four-body forces, and not in any sense from the $U(\rho)$

[6] In this respect wedges of perfectly reflecting materials behave similarly: Brevik and Lygren (1996) calculate that the *outward* stress normal to the surface is proportional to $\left[(\pi/\Phi)^2 + 11\right]\left[1 - (\pi/\Phi)^2\right]$, i.e. positive for concave and negative for convex wedges (their α is our Φ). Their full expressions agree with those of Deutsch and Candelas (1979), but differ from those of Lukosz (1973), who claims moreover that wedges tend to open rather than to close.

acting twice. The assumption is one that our model shares with all treatments of Casimir effects for continua. By contrast, in real (granular) materials the operative correlation function itself becomes dependent on the two-body forces, a dependence that is then transmitted to the analogue of gU, causing the energy B to acquire terms of second and of higher orders in U. This paper allows for such terms only indirectly, insofar as they validate the cutoff λ.

3.2. Proof

Let (in) specify the interior and (out) the exterior of the body, so that

$$(\text{in}) + (\text{out}) = (\text{all space}).$$

Recall that the system called the *cavity* is defined as the region (out) filled with medium, plus the region (in) empty.

The infinite energy B^* can be written formally as

$$\begin{aligned} B^* = V^*u &= \tfrac{1}{2}\int_{(\text{in})+(\text{out})} \mathrm{d}^3 r \int_{(\text{in})+(\text{out})} \mathrm{d}^3 r'\, U(\boldsymbol{r}-\boldsymbol{r}') \\ &= \tfrac{1}{2}(\text{in},\text{in}) + \tfrac{1}{2}(\text{out},\text{out}) + (\text{in},\text{out}) \end{aligned} \tag{3.5}$$

using the obvious notation

$$(\text{in},\text{in}) \equiv \int_{(\text{in})} \mathrm{d}^3 r \int_{(\text{in})} \mathrm{d}^3 r'\, U(\boldsymbol{r}-\boldsymbol{r}')$$

and so on. B^* diverges because (out, out) diverges.

From its definition, u is independent of position. However, focusing on some arbitrary point $\boldsymbol{r}$ of (in), we see that

$$u(\boldsymbol{r}) = u = \tfrac{1}{2}\left\{\int_{(\text{in})} \mathrm{d}^3 r'\, U(\boldsymbol{r}-\boldsymbol{r}') + \int_{(\text{out})} \mathrm{d}^3 r'\, U(\boldsymbol{r}-\boldsymbol{r}')\right\}. \tag{3.6}$$

Integration over (in) yields

$$Vu = \tfrac{1}{2}(\text{in},\text{in}) + \tfrac{1}{2}(\text{in},\text{out}). \tag{3.7}$$

Note the factor $1/2$ of the second term.

For the energies B_b and B_c one has

$$B_\mathrm{b} = \tfrac{1}{2}(\text{in},\text{in}) \equiv Vu + S\sigma + \Delta B_\mathrm{b} \tag{3.8}$$

$$B_\mathrm{c} = \tfrac{1}{2}(\text{out},\text{out}) \tag{3.9}$$

whence

$$\begin{aligned} B_\mathrm{c} - B^* &= \tfrac{1}{2}(\text{out},\text{out}) - \left\{\tfrac{1}{2}(\text{in},\text{in}) + \tfrac{1}{2}(\text{out},\text{out}) + (\text{in},\text{out})\right\} \\ &= -\tfrac{1}{2}(\text{in},\text{in}) - (\text{in},\text{out}) \equiv -Vu + S\sigma + \Delta B_\mathrm{c}. \end{aligned} \tag{3.10}$$

Eliminating (in, out) and (in, in) from (3.7), (3.8) and (3.10) we find

$$\begin{aligned} -Vu + S\sigma + \Delta B_\mathrm{c} &= -\tfrac{1}{2}(\text{in},\text{in}) - \{2Vu - (\text{in},\text{in})\} = \tfrac{1}{2}(\text{in},\text{in}) - 2Vu \\ &= \{Vu + S\sigma + \Delta B_\mathrm{b}\} - 2Vu = -Vu + S\sigma + \Delta B_\mathrm{b} \Rightarrow \Delta B_\mathrm{c} = \Delta B_\mathrm{b}. \end{aligned} \tag{3.11}$$

□

We stress that the theorem holds for any translation-invariant and symmetric potential $U(\boldsymbol{r},\boldsymbol{r}') = U(\boldsymbol{r}',\boldsymbol{r})$ for which the integrals (in, in) and (in, out) converge, and for bodies and holes of any shape.

4. Renormalization

By renormalization we mean *only* that the components $(\pm Vu + S\sigma)$ of the total energy are identified and displayed separately. All other components are assigned to the *renormalized energy* ΔB: sections 6–11 will show that they can be of very different kinds.

We call (nominally) *divergent* those expressions that, like u and σ, would diverge in the wholly hypothetical limit $\lambda \to 0$. They include s–d terms, which tend to distort simple shapes rather than to change sizes. *Pure Casimir terms* are defined as convergent (hence independent of λ) and independent of Ω; for finite bodies they must on dimensional grounds be inversely proportional to linear size parameters, for example to $1/a$ with a the radius of a sphere or the edge-length of a cube. For parallel-sided slabs, ΔB consists wholly of pure Casimir terms; for spheres it contains divergent s–d terms as well; for cubes, the edges contribute to ΔB further divergent terms proportional to $a\Omega/\lambda$, which are not absorbed by renormalization but do nevertheless depend on size as well as on shape. Likewise, ΔB per unit length of an infinitely long right circular cylinder of radius a is proportional to $(\Omega/a)\log(1/2\Omega\lambda)$, entangling a, Ω and λ so that no simple prescription and certainly no power-series in λ can unravel them.

Here it bears repeating, from section 1, that one cannot afford to ignore *any* components of B, convergent or divergent. Attempts to discard divergent components (often sight unseen) probably stem from confusing physically appropriate *renormalization* with the purely mathematical tools of dimensional or zeta-function *regularization* familiar elsewhere in field theory.

Finally it may be worth spelling out why, in contrast to $(\pm Vu + S\sigma)$, we assign the edge-governed divergences to the renormalized component ΔB. The reason is simply that condensed-state physics regards u and σ as standard material properties, but concedes no such status to $\zeta(\Phi)$. Admittedly it remains a matter of taste whether this is regarded as a difference of principle or merely of practice; but in practice it would certainly be bewildering to introduce a renormalization scheme with infinitely many different counterterms to allow for any of the infinitely many different opening angles Φ that sharp edges can have. Similar arguments apply to s–d terms.

5. The correlation function $g(\rho)$

5.1. Definition

It proves convenient to express the total binding energy as a convolution of the potential with another function determined purely by the geometry of the body:

$$B = -(n\alpha)^2\tfrac{1}{2}\iint \mathrm{d}^3r\,\mathrm{d}^3r'\,\chi(\boldsymbol{r})\chi(\boldsymbol{r}')f(\rho)$$

$$= -(n\alpha)^2\tfrac{1}{2}\iint \mathrm{d}^3r\,\mathrm{d}^3r'\int \mathrm{d}^3\rho\,\delta(\boldsymbol{r}'-\boldsymbol{r}-\boldsymbol{\rho})\chi(\boldsymbol{r})\chi(\boldsymbol{r}')f(\rho)$$

$$B = -(n\alpha)^2\tfrac{1}{2}\int \mathrm{d}^3\rho\,f(\rho)\tilde{g}(\boldsymbol{\rho}) = -(n\alpha)^2\tfrac{1}{2}\int_\lambda^\infty \mathrm{d}\rho\,f(\rho)\rho^2 g(\rho) \tag{5.1}$$

where we have defined an auxiliary function

$$\tilde{g}(\boldsymbol{\rho}) \equiv \iint \mathrm{d}^3r\,\mathrm{d}^3r'\,\chi(\boldsymbol{r})\chi(\boldsymbol{r}')\delta(\boldsymbol{r}'-\boldsymbol{r}-\boldsymbol{\rho}) \tag{5.2}$$

and the two-point *correlation function*

$$g(\rho) \equiv \int \mathrm{d}\Omega_\rho\,\tilde{g}(\boldsymbol{\rho}) = \iint \mathrm{d}^3r\,\mathrm{d}^3r'\,\chi(\boldsymbol{r})\chi(\boldsymbol{r}')\int \mathrm{d}\Omega_\rho\,\delta(\boldsymbol{r}'-\boldsymbol{r}-\boldsymbol{\rho}). \tag{5.3}$$

The best way to $\tilde{g}$ is often through the *form factor* F of the body:

$$\tilde{g}(\boldsymbol{\rho}) \equiv \int \frac{\mathrm{d}^3k}{(2\pi)^3} \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\rho})\,|F(\boldsymbol{k})|^2\,, \qquad F(\boldsymbol{k}) \equiv \int \mathrm{d}^3r\,\chi\,(\boldsymbol{r}) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}). \tag{5.4}$$

Notice $[\tilde{g}] = [g] = [L]^3$. There is an obvious pictorial interpretation of (5.3): draw a spherical shell of radius ρ, centred on an interior point $\boldsymbol{r}$; determine the solid angle subtended at $\boldsymbol{r}$ by that part of the shell that is inside the body; and integrate the result over the interior with respect to d^3r.

5.2. Examples

For unbounded space, the definition of g as verbalized just above yields

$$g(\rho) = 4\pi \int \mathrm{d}^3r = 4\,\pi\,V^* \quad \text{(unbounded space)}. \tag{5.5}$$

For the half-space $z > 0$, it yields

$$\begin{aligned} g(\rho) &= \iint \mathrm{d}x\,\mathrm{d}y \int_0^\infty \mathrm{d}z\,\{\Theta(z-\rho)4\pi + \Theta(\rho-z)2\pi(1+z/\rho)\} \\ &= S\left\{4\pi\int_0^\infty \mathrm{d}z - \pi\rho\right\} \\ g(\rho) &= 4\pi\,V - \pi\,S\rho, \qquad S = A \equiv \iint \mathrm{d}x\,\mathrm{d}y \quad \text{(half-space)} \end{aligned} \tag{5.6}$$

where S is the surface area, and $V = A\int_0^\infty \mathrm{d}z = V^*/2$ the volume. The Heaviside step-function is defined as

$$\Theta(z>0) = 1 \qquad \Theta(z<0) = 0. \tag{5.7}$$

By contrast, for the slab $-a/2 < z < a/2$ it is already better to use (5.4) with

$$F(\boldsymbol{k}) = (2\pi)^2\delta(k_x)\delta(k_y)2\sin(k_za/2)/k_z, \qquad \left[(2\pi)^2\delta(k_x)\delta(k_y)\right]^2 = A(2\pi)^2\delta(k_x)\delta(k_y).$$

In terms of $\kappa \equiv k_z$ and of $\cos\theta \equiv \rho_z/\rho$ this leads to

$$\begin{aligned} \tilde{g}(\boldsymbol{\rho})/A &= \frac{2}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\kappa\,\exp(-\mathrm{i}\kappa\rho\cos\theta)\frac{\sin^2(\kappa a/2)}{\kappa^2} \\ g(\rho)/A &= \int_{-1}^{1} 2\pi\,\mathrm{d}\cos\,\theta\,\tilde{g}(\boldsymbol{\rho}) = \frac{8}{\rho}\int_{-\infty}^{\infty}\mathrm{d}\kappa\,\frac{\sin(\kappa\rho)\sin^2(\kappa a/2)}{\kappa^3} \\ &= \Theta(a-\rho)\pi(4a-2\rho) + \Theta(\rho-a)\frac{2\pi a^2}{\rho} \\ g(\rho) &= \Theta(a-\rho)\,(4\pi\,V - \pi\,S\rho) + \Theta(\rho-a)A\frac{2\pi a^2}{\rho} \quad \text{(slab of width } a) \end{aligned} \tag{5.8}$$

where we have used $V = Aa$ and $S = 2A$. Though singular at $\rho = a$, g is continuous: $g(a-0) = g(a+0)$.

Finally, for a sphere of radius a one can find $g(\rho)$ either through an elementary integration as in appendix A of I, or through its form factor:

$$F(\boldsymbol{k}) = \int \mathrm{d}^3r\,\Theta(a-r)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) = \frac{4\pi}{k}\left[\frac{\sin(ka) - ka\cos(ka)}{k^2}\right] \tag{5.9}$$

$$g(\rho) \equiv \int \mathrm{d}\Omega_\rho \int \frac{\mathrm{d}^3k}{(2\pi)^3} \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\rho})\,|F(\boldsymbol{k})|^2$$
$$= \frac{32\pi}{\rho}\int_0^\infty \frac{\mathrm{d}k}{k^5}\sin(k\rho)\,[\sin(ka) - ka\cos(ka)]^2$$
$$= \Theta(2a-\rho)\pi^2\left\{\frac{16a^3}{3} - 4a^2\rho + \frac{\rho^3}{3}\right\}$$
$$g(\rho) = \Theta(2a-\rho)\left\{4\pi V - \pi S\rho + \frac{\pi^2\rho^3}{3}\right\} \quad \text{(sphere of radius } a\text{)}. \tag{5.10}$$

Thus g is given by its Taylor series multiplied by a step function, making calculations far simpler for the sphere than for finite bodies of any other shape.

5.3. The Taylor series

Write the series as

$$g(\rho) = g_0 + g_1\rho + g_2\rho^2/2! + g_3\rho^3/3! + \cdots. \tag{5.11}$$

For arbitrary regions with smooth surfaces, or with edges but no vertices, appendix C shows that the leading coefficients are determined explicitly by the volume V of the region and by the *local* geometry of its surface S (whose area we are likewise calling S).

First of all one finds, quite generally,

$$g_0 = 4\pi V \qquad g_1 = -\pi S \tag{5.12}$$

just as the examples in section 5.2 might have suggested. The next coefficient is nonzero only if the body has sharp edges:

$$g_2 = \tfrac{4}{3}\int \mathrm{d}L\,\zeta(\Phi(L)) \tag{5.13}$$

where ζ is given by (2.17), and the integral runs along all such edges, with $\mathrm{d}L$ the element of edge length and $\Phi(L)$ the angle between the local tangent planes (see figure C.1). Lastly

$$g_3 = \frac{\pi}{16}\int_S \mathrm{d}S\left[\frac{3}{R_1^2} + \frac{2}{R_1R_2} + \frac{3}{R_2^2}\right] \quad \text{(smooth surface)} \tag{5.14}$$

where $R_{1,2}$ are the principal radii of curvature. This expression applies only if $R_{1,2} \gg \lambda$, so that one cannot find the contributions of edges and vertices by taking limits $R_i \to 0$. In fact the cube (section 10) shows that vertices make additional contributions to g_3.

Notice that g_3 is *not* determined by topology alone: the Gauss–Bonnet theorem (Coxeter 1969) reads $\int_S \mathrm{d}S/R_1R_2 = 2\pi\chi_\mathrm{E}$, where χ_E is the Euler characteristic of the surface (e.g. 2 for a sphere and 0 for a torus).

5.4. Binding energies

For unbounded space, (5.1) and (5.5) immediately reproduce $B = V^*u$ with u from (2.15); similarly, for a half-space, (5.1) and (5.6) reproduce $B = (V^*/2)u + S\sigma$ with σ from (2.16). But for finitely bounded bodies the correlation function has singularities at finite values of ρ, so that it is not fully specified by its Taylor series alone. Nevertheless, by substituting (5.11) into (5.1) we can identify all the divergent components of B, because they are governed by the behaviour of fg as $\rho \to \lambda \to 0$, where the series does represent g. The substitution yields

$$B \simeq -(n\alpha)^2\frac{1}{2}\left\{g_0\mathcal{J}_2(\lambda) + g_1\mathcal{J}_3(\lambda) + \frac{1}{2!}g_2\mathcal{J}_4(\lambda) + \frac{1}{3!}g_3\mathcal{J}_5(\lambda) + \cdots\right\}$$

$$= -(n\alpha)^2\frac{1}{2}\left\{[4\pi V\mathcal{J}_2(\lambda)] - [\pi S\mathcal{J}_3(\lambda)] + \left[\frac{1}{2!}\frac{4}{3}\int \mathrm{d}L\, Z(\Phi(L))\mathcal{J}_4(\lambda)\right]\right.$$
$$\left. +\frac{1}{3!}\left[\frac{\pi}{16}\int \mathrm{d}S\left(\frac{3}{R_1^2}+\frac{2}{R_1R_2}+\frac{3}{R_2^2}\right) + (\text{vertex contributions})\right]\mathcal{J}_5(\lambda) + \cdots\right\}. \tag{5.15}$$

To exhibit the successive orders of magnitude, we estimate each moment by its leading term, and the coefficients from the typical linear dimensions a of a (macroscopic) body or cavity, with $V \sim a^3$, $S \sim a^2$, $\int \mathrm{d}L\, Z \sim a$ and $\int \mathrm{d}S/R^2 \sim a^0$. This yields

$$B \sim (n\alpha)^2\Omega\left\{\left[\mathcal{O}\left(\frac{a}{\lambda}\right)^3\right] + \left[\mathcal{O}\left(\frac{a}{\lambda}\right)^2\right] + \left[\mathcal{O}\left(\frac{a}{\lambda}\right) \text{ if there are edges}\right]\right.$$
$$+\mathcal{O}\left[\log\left(\frac{1}{\Omega\lambda}\right)\right]\times\left[\begin{pmatrix}\text{explicitly known terms,}\\ \text{of } \mathcal{O}(1), \text{ from curved surfaces}\end{pmatrix}\right.$$
$$\left.\left. +\begin{pmatrix}\text{terms not generally known,}\\ \text{of } \mathcal{O}(1), \text{ from vertices}\end{pmatrix}\right] + \cdots\right\} \tag{5.16}$$

with each $[\cdots]$ smaller than the one before by the minuscule ratio λ/a of atomic λ to macroscopic a; we reckon the logarithm, realistically, as of order unity. Pure Casimir energies, not indicated above, depart from this pattern: being of order $(n\alpha)^2/a$, their ratio to the last $[\cdots]$ in (5.16) is of order $1/\Omega a \sim$ (excitation wavelength) $/a \gg \lambda/a$.

Notice finally that B has no terms linear in the $1/R_i$, i.e. none proportional to $(\Omega/\lambda)\int_S \mathrm{d}S\,(1/R_1 + 1/R_2)$. This was foreseeable: they would violate the cavity theorem, because for body and cavity the R_i have opposite signs. Smooth surfaces produce such contributions only to higher-order correlation functions, relevant to the energy only to higher than second order in $(n\alpha)$. This is illustrated by the component of ΔB that Candelas[7] (1982, equation (1.10)) writes as $\mathcal{E}^{\mathrm{C}}\int_S \mathrm{d}S\,(1/R_1 + 1/R_2) \sim \mathcal{E}^{\mathrm{C}}a$. Using his (5.38), (5.40) and (5.41), one can expand his $\mathcal{E}^{\mathrm{C}}$ in powers of $n\alpha$. Then the terms of order $n\alpha$ and $(n\alpha)^2$ cancel in a way which in that context appears quite fortuitous: one finds $\mathcal{E}^{\mathrm{C}} \simeq (\pi^2/32)(n\alpha)^3 k_{\mathrm{C}}\Omega$, with k_{C} a wavenumber cutoff, comparable perhaps to our $1/\lambda$. Candelas's conclusion could be incorporated into (5.16) by amending the third pair of square brackets to

$$\mathcal{O}\left(\frac{a}{\lambda}\right)\times\left[\begin{pmatrix}\text{explicitly known terms}\\ \text{of } \mathcal{O}(1) \text{ from edges}\end{pmatrix} + \begin{pmatrix}\text{unknown terms of } \mathcal{O}(n\alpha)\\ \text{from smooth surfaces}\end{pmatrix}\right]. \tag{5.17}$$

Admittedly, the correction of $\mathcal{O}(n\alpha)$ in (5.17) might well exceed the entire next term in the sequence (5.16). Their ratio is of order $(n\alpha)\,a/\lambda$; but for solid argon for instance appendix F suggests $n\alpha \sim 0.04$, likely to be overcompensated by a/λ. The same caution applies to possible corrections of relative order $(n\alpha)$ to the other entries in (5.15) and (5.16). However, here one is comparing terms that are not in competition, because they vary differently with a.

6. Gap and slab

These are related as cavity to body. Consider a gap of width a, with parallel sides each of infinite area A, so that $V = Aa$ and $S = 2A$. We deal with it *ad hoc*, the better to motivate the systematic renormalization procedure developed in the next section a propos of spheres.

The total energy B is given by (5.1) with (5.8). For $a > \lambda$ they yield

$$B/A = -(n\alpha)^2\frac{\pi}{2}\left\{\int_\lambda^a \mathrm{d}\rho\, f(4a\rho^2 - 2\rho^3) + \int_a^\infty \mathrm{d}\rho\, f 2a^2\rho\right\}$$

[7] Candelas's ω_{E}, ω_{p}, β_{E}, α_{E} are our Ω, $\Omega\sqrt{4\pi n\alpha}$, $\Omega\sqrt{1+4\pi n\alpha}$, $\Omega\sqrt{1+2\pi n\alpha}$. His cutoff procedure is completely different from ours, so that it would make little sense to compare coefficients in detail.

$$= -(n\alpha)^2\left\{\left[2\pi a\mathcal{J}_2(\lambda)-\pi\mathcal{J}_3(\lambda)\right]+\pi\left[\mathcal{J}_3(a)-2a\mathcal{J}_2(a)+a^2\mathcal{J}_1(a)\right]\right\}. \quad (6.1)$$

Inspection shows that the contribution from the first pair of square brackets is just $-Vu+S\sigma$, whence

$$\Delta B(a)/A = -(n\alpha)^2\,\pi\left\{\mathcal{J}_3(a)-2a\mathcal{J}_2(a)+a^2\mathcal{J}_1(a)\right\} \qquad (a>\lambda). \quad (6.2)$$

Thus ΔB contains nothing but pure Casimir terms; unfortunately gap and slab seem to be unique in this respect.

In the VdW regime, substitution from (2.10)–(2.12) into (6.2) yields

$$\Delta B(a)/A \simeq -(n\alpha)^2\,\frac{\pi\Omega}{16a^2}\left\{1-4(\Omega a)^2\left[\log\left(\frac{1}{2\Omega a}\right)-\gamma-\frac{3}{4}\right]+\mathcal{O}\left[(\Omega a)^3\log\left(\frac{1}{2\Omega a}\right)\right]\right\} \quad (6.3)$$

whose first term is well known. In the CP regime the $\mathcal{J}_N(a)$ are evaluated using (2.6):

$$\Delta B(a)/A \simeq -(n\alpha)^2\,\frac{1}{a^3}\left\{\frac{23}{120}-\frac{43}{140}\frac{1}{(\Omega a)^2}+\mathcal{O}\left[\frac{1}{(\Omega a)^4}\right]\right\}. \quad (6.4)$$

Perfect reflectors have $\Delta B(a)/A = -\pi^2/720a^3$. For a crude and quite unwarranted comparison one might substitute into the leading term of (6.4) the utterly nonperturbative value $n\alpha = 3/4\pi = 0.239$, which according to the Clausius–Mossotti formula entails $\varepsilon\to\infty$. This would turn the leading coefficient into $(23/120)(3/4\pi)^2 = 0.011$, whose proximity to $\pi^2/720 = 0.014$ is probably fortuitous.

As the gap closes with $a\to 0 \Rightarrow a<\lambda$, the step function $\Theta(a-\rho)$ in g vanishes, leaving one with

$$B/A = -(n\alpha)^2\,\frac{\pi}{2}\int_\lambda^\infty \mathrm{d}\rho\, f 2a^2\rho = -(n\alpha)^2\,\pi a^2\mathcal{J}_1(\lambda);$$

then the definition $\Delta B \equiv B+Vu-S\sigma = B+aAu-2A\sigma$ yields

$$\Delta B(a)/A = -2\sigma+au-a^2\,(n\alpha)^2\,\pi\mathcal{J}_1(\lambda) \qquad (a<\lambda). \quad (6.5)$$

Thus $\Delta B(0)/A = -2\sigma$; this is what one expects, because two half-spaces with a zero-width gap between them constitute unbounded space, whence their mutual interaction must identically cancel their surface energies.

7. Sphere

Equations (5.1) and (5.10) yield

$$B = -\frac{1}{2}(n\alpha)^2\int_\lambda^\infty \mathrm{d}\rho\,\rho^2 f(\rho)g(\rho) = -(n\alpha)^2\,\frac{\pi^2}{2}\int_\lambda^{2a}\mathrm{d}\rho\, f\rho^2\left\{\frac{16a^3}{3}-4a^2\rho+\frac{\rho^3}{3}\right\}. \quad (7.1)$$

To prepare B for renormalization we write

$$\int_\lambda^{2a}\mathrm{d}\rho\, f\rho^N = \left\{\int_\lambda^\infty \mathrm{d}\rho-\int_{2a}^\infty \mathrm{d}\rho\right\} f\rho^N = \mathcal{J}_N(\lambda)-\mathcal{J}_N(2a). \quad (7.2)$$

Then

$$B = -(n\alpha)^2\left\{\left[\frac{8\pi^2a^3}{3}\mathcal{J}_2(\lambda)-2\pi^2a^2\mathcal{J}_3(\lambda)\right]+\frac{\pi^2}{6}\mathcal{J}_5(\lambda)\right.$$
$$\left.-\frac{8\pi^2a^3}{3}\mathcal{J}_2(2a)+2\pi^2a^2\mathcal{J}_3(2a)-\frac{\pi^2}{6}\mathcal{J}_5(2a)\right\}; \quad (7.3)$$

the square brackets contribute precisely $Vu + S\sigma$, whence

$$\Delta B = -(n\alpha)^2 \frac{\pi^2}{6}\mathcal{J}_5(\lambda) + (n\alpha)^2\pi^2 \left\{ \frac{8a^3}{3}\mathcal{J}_2(2a) - 2a^2\mathcal{J}_3(2a) + \frac{1}{6}\mathcal{J}_5(2a) \right\}. \tag{7.4}$$

The leading term, with $\mathcal{J}_5(\lambda)$, is shape-dependent: in particular it survives unchanged in the limit $a \to \infty$. The other three are pure Casimir terms; for macroscopic $a \gg 1/\Omega$, they can be approximated by evaluating the $\mathcal{J}_N(2a)$ with the CP expansions (2.6) for f, as in deriving (6.4). This leads to

$$\Delta B \simeq -(n\alpha)^2 \frac{\pi^2\Omega}{8} \left\{ \log\left(\frac{1}{2\Omega\lambda}\right) - \gamma + \frac{65}{24} \right\} + (n\alpha)^2 \frac{\pi}{a} \left\{ \frac{23}{96} - \frac{43}{960}\frac{1}{(\Omega a)^2} + \cdots \right\}. \tag{7.5}$$

Nondispersive models naturally produce quite different expressions. For instance, (7.1) with $f \to 23/4\pi\rho^7$ for all $\rho > \lambda$ yields

$$B = (n\alpha)^2 \left\{ -\frac{23}{8\lambda^4}V + \frac{23}{24\lambda^3}S - \frac{23\pi}{24\lambda} + \frac{23\pi}{96a} \right\} \quad \text{(nondispersive, cutoff } \rho > \lambda) \tag{7.6}$$

as already reported by Marachevsky (2000). Alternatively, one might as in I adopt a different nondispersive potential $f(\rho, \tilde{\lambda})$, evaluated with an exponential cutoff $\exp(-\tilde{\lambda}k)$ in Fourier space. This yields the expressions quoted in (E.6) and (E.7), which entail[8]

$$B = (n\alpha)^2 \left\{ -\frac{3}{8\tilde{\lambda}^4}V + \frac{14\pi}{45\tilde{\lambda}^3}S - \frac{4}{5\tilde{\lambda}} + \frac{23\pi}{96a} \right\} \quad \text{(nondispersive, cutoff } \exp(-\tilde{\lambda}k)). \tag{7.7}$$

For a perfectly reflecting and infinitesimally thin spherical shell the pure Casimir term, by general consent, is $\simeq 0.092/a$. Derived by Boyer (1968) and confirmed by Davies (1972), it has remained under study ever since: see e.g. Candelas (1982), Lambiase *et al* (1999), Leseduarte and Romeo (1996), Bordag *et al* (1999), Klich (2000), Hagen (2000a) and references there, Brevik *et al* (2000), Hagen (2000b) and Esposito *et al* (2000). The presence of a negative shape-dependent divergence is emphasized by Candelas (1982), albeit he makes it linear instead of logarithmic. Of course it is unclear just what physics one can learn by comparing particular contributions to the energy of a shell with *prima facie* similar contributions to the energy of a solid sphere or of a spherical cavity in an otherwise unbounded medium. Ignoring such scruples, one might make the same crude comparison as in section 6: with $n\alpha = 3/4\pi$ the pure Casimir term becomes $69/512\pi a = 0.043/a$.

8. Finitely thin spherical shell

Consider a material shell with thickness $t \ll R$ and radii $b = R + t/2$ and $a = R - t/2$.

By hindsight, start with a spherical cavity of radius b around a concentric solid sphere of radius a. We adapt in an obvious way the notation of section 3.2 for double volume integrals, and write the volume and surface of spheres with radii a, b as $V(a)$, $S(a)$ and $V(b)$, $S(b)$ respectively. (Do not confuse the outer radius b with the suffix b for 'body'.) Then, in view of (3.3),

$$\begin{aligned} B_{\mathrm{c}}(\text{shell}) &= \tfrac{1}{2}(\rho > b, \rho > b) + \tfrac{1}{2}(\rho < a, \rho < a) + (\rho < a, \rho > b) \\ &\equiv -V(\text{shell})u + S(\text{shell})\sigma + \Delta B_{\mathrm{c}}(\text{shell}) + B^*. \end{aligned} \tag{8.1}$$

[8] Before comparing expressions like (7.6) or (7.7) with (7.4) and (7.5), one would need to decide whether, in nondispersive models, cutoff lengths should continue to be understood as of the order of the lattice spacing, or whether they should be re-interpreted as of the order of the much longer optical wavelengths $1/\Omega$. For reasons apparent in particular from appendix E.1, the writer thinks that the question is empty, because nondispersive models are unreliable guides to the physics of connected bodies in the first place.

Similarly

$$\tfrac{1}{2}(\rho > b, \rho > b) = B_{\rm c}(\text{sphere, radius } b) + B^* \equiv -V(b)u + S(b)\sigma + \Delta B_{\rm c}(\text{sphere, radius } b) + B^* \tag{8.2}$$

$$\tfrac{1}{2}(\rho < a, \rho < a) = B_{\rm b}(\text{sphere, radius } a) \equiv V(a)u + S(a)\sigma + \Delta B_{\rm b}(\text{sphere, radius } a). \tag{8.3}$$

Combining (8.1)–(8.3) we note $V(\text{shell}) = V(b) - V(a)$ and $S(\text{shell}) = S(b) + S(a)$, and by virtue of the cavity theorem $\Delta B_{\rm c} = \Delta B_{\rm b}$ write the result as

$$\Delta B(\text{shell}) = \Delta B(\text{sphere, radius } a) + \Delta B(\text{sphere, radius } b) + (\rho < a, \rho > b). \tag{8.4}$$

From here on we consider only the CP regime $t\Omega \gg 1$. Then the final integral on the right may be approximated by $f(\rho) \to -C_{\rm CP}/\rho^7$:

$$(\rho < a, \rho > b) \simeq -C_{\rm CP}\int_{r<a} {\rm d}^3 r \int_{r>b} {\rm d}^3 r' \, \frac{1}{|\,\boldsymbol{r} - \boldsymbol{r}'\,|^7} = -C_{\rm CP}\frac{4\pi^2}{15}\frac{a^3(5b^2 - a^2)}{(b^2 - a^2)^3}. \tag{8.5}$$

Whether the shell tends to expand or to collapse depends on its *total* energy $B_{\rm b}$, which we may now approximate by dropping, from terms of given order in $1/\lambda$, corrections of relative order $1/\Omega a$ etc:

$$\frac{B_{\rm b}(\text{shell})}{(n\alpha)^2} \simeq -\frac{4\pi(b^3 - a^3)}{3}\frac{\pi\Omega}{2\lambda^3} + 4\pi(b^2 + a^2)\frac{3\pi\Omega}{16\lambda^2} - \frac{\pi^2\Omega}{3}\left[\frac{3}{4}\log\left(\frac{1}{2\Omega\lambda}\right) - \frac{3\gamma}{4} + \frac{65}{32}\right] + \frac{23\pi}{4}\left[\frac{1}{24}\left(\frac{1}{b} + \frac{1}{a}\right) - \frac{4}{15}\frac{a^3(5b^2 - a^2)}{(b^2 - a^2)^3}\right]. \tag{8.6}$$

Finally, expansion in powers of t/R yields

$$\frac{B_{\rm b}(\text{shell})}{\pi^2(n\alpha)^2} \simeq -\frac{\Omega}{\lambda^3}\left[2R^2 t + \frac{t^3}{6}\right] + \frac{\Omega}{\lambda^2}\left[\frac{3R^2}{2} + \frac{3t^2}{8}\right] - \Omega\left[\frac{1}{4}\log\left(\frac{1}{2\Omega\lambda}\right) - \frac{\gamma}{4} + \frac{65}{96}\right] + \frac{23}{4\pi}\left[-\frac{2R^2}{15t^3} + \frac{1}{6t} + \mathcal{O}\left(\frac{t^2}{R^3}\right)\right]. \tag{8.7}$$

The component proportional to R^2/t^3 is just the pure Casimir energy of a slab or gap of width t and total area $4\pi R^2$; but internal cancellations have removed any visible reminder of the $1/R$-proportional pure Casimir energies of spheres with radii near R, and there is no obvious interpretation of the much larger term proportional to $1/t$.

9. Pressure on infinitesimally thin spherical shell

Consider the further idealization of a spherical shell with radius R and thickness $t \to 0 \Rightarrow t < \lambda$ (sic), but with fixed finite $\Gamma \equiv 4\pi(n\alpha)t$ (so that $[\Gamma] = [L]$). We calculate directly the net pressure P experienced by the shell, incidentally sidestepping the difficulties, outlined in section 2, of deriving stresses from energies. Along the same lines as in appendix D of I, one finds

$$P = \left(\frac{\Gamma}{4\pi}\right)^2 \frac{\pi}{R}\big\{-2\mathcal{J}_1(\lambda) + (2R)^2 f(2R) + 2\mathcal{J}_1(2R)\big\},$$

reckoning outward pressure as positive. In the CP regime $R\Omega \gg 1$ this reduces to

$$P \simeq \left(\frac{\Gamma}{4\pi}\right)^2 \left\{-\frac{3\pi}{8}\frac{\Omega}{\lambda^4 R} + \frac{161}{640}\frac{1}{R^6}\right\}.$$

Again P is attractive, while the pure Casimir term is repulsive.

10. Cube

Consider a cube with edge length a. Its form factor is $F(\boldsymbol{k}) = F_1F_2F_3$, where $F_i = (2/k_i)\sin(k_ia/2)$; then (5.4) yields

$$\tilde{g} = \Theta\,(a- \mid \rho_1 \mid)\,(a- \mid \rho_1 \mid) \times \Theta\,(a- \mid \rho_2 \mid)\,(a- \mid \rho_2 \mid) \times \Theta\,(a- \mid \rho_3 \mid)\,(a- \mid \rho_3 \mid)\,.$$

Thus the integration in $g(\rho) \equiv \int \mathrm{d}\Omega_\rho\, \tilde{g}(\boldsymbol{\rho})$ runs over part of the surface of a sphere of radius ρ, namely the part that lies inside a concentric cube in ρ-space having edges of length $2a$. For $\rho < a$ all of the sphere contributes; for $a < \rho < a\sqrt{2}$ one must subtract the portions protruding from each face; for $a\sqrt{2} < \rho < a\sqrt{3}$ this correction must itself be corrected for overlaps between the protrusions across adjacent faces; while for $\rho > a\sqrt{3}$ the sphere lies wholly outside the cube, so that g vanishes. Painfully, one obtains

$$g = \begin{cases} g^{(1)} & \text{for} \quad \rho < a \\ g^{(2)} & \text{for} \quad a < \rho < a\sqrt{2} \\ g^{(3)} & \text{for} \quad a\sqrt{2} < a\sqrt{3} \\ 0 & \text{for} \quad \rho > a\sqrt{3} \end{cases} \tag{10.1}$$

$$g^{(1)} = \left\{4\pi a^3 - 6\pi a^2\rho + 8a\rho^2 - \rho^3\right\} \tag{10.2}$$

$$g^{(2)} = \left\{24a^2\rho\cos^{-1}\left(\frac{a}{\rho}\right) - \frac{8a}{\rho}\left(a^2+2\rho^2\right)\sqrt{\rho^2-a^2} + \frac{\pi}{\rho}\left(6a^4 - 8a^3\rho\right) + \frac{1}{\rho}\left(-a^4 + 6a^2\rho^2 + 2\rho^4\right)\right\} \tag{10.3}$$

$$g^{(3)} = \left\{-24\frac{a^2}{\rho}(a^2+\rho^2)\cos^{-1}\left(\frac{a}{\sqrt{\rho^2-a^2}}\right) - 24a^3\tan^{-1}\left(\frac{a^2}{\rho\sqrt{\rho^2-2a^2}}\right) + \frac{8a}{\rho}\left(a^2+\rho^2\right)\sqrt{\rho^2-2a^2} + \frac{\pi}{\rho}\left(6a^4 + 4a^3\rho + 6a^2\rho^2\right) - \frac{1}{\rho}\left(5a^4 + 6a^2\rho^2 + \rho^4\right)\right\}. \tag{10.4}$$

In $g^{(1)}$ the first three terms are dictated through the Taylor series by the volume $V = a^3$, the surface area $S = 6a^2$ and total edge length $L = 12a$ (with opening angle $\Phi = \pi/2$). By contrast to (5.10) for the sphere, the shape-dependent term $-\rho^3$ is now negative; presumably it is governed just by the vertices, but the writer has not managed to determine the contribution from vertices of arbitrary shape.

We write down B and renormalize it without further comment:

$$B = -(n\alpha)^2\tfrac{1}{2}\int_\lambda^{a\sqrt{3}} \mathrm{d}\rho\; f\rho^2 g$$

$$B = -(n\alpha)^2\tfrac{1}{2}\left\{\int_\lambda^\infty \mathrm{d}\rho\; f\rho^2 g^{(1)} - \int_a^\infty \mathrm{d}\rho\; f\rho^2 g^{(1)} + \int_a^{a\sqrt{2}} \mathrm{d}\rho\; f\rho^2 g^{(2)} + \int_{a\sqrt{2}}^{a\sqrt{3}} \mathrm{d}\rho\; f\rho^2 g^{(3)}\right\}$$

$$= -(n\alpha)^2\tfrac{1}{2}\left\{\left[4\pi a^3\mathcal{J}_2(\lambda) - 6\pi a^2\mathcal{J}_3(\lambda)\right] + \left[8a\mathcal{J}_4(\lambda) - \mathcal{J}_5(\lambda)\right] + \left[-\int_a^\infty \mathrm{d}\rho\; f\rho^2 g^{(1)} + \int_a^{a\sqrt{2}} \mathrm{d}\rho\; f\rho^2 g^{(2)} + \int_{a\sqrt{2}}^{a\sqrt{3}} \mathrm{d}\rho\; f\rho^2 g^{(3)}\right]\right\}. \tag{10.5}$$

The first pair of square brackets contributes $Vu + S\sigma$; the contents of the second pair depend on both a and λ; the third contribute the pure Casimir terms. Accordingly

$$\Delta B = -(n\alpha)^2\tfrac{1}{2}\left\{8a\mathcal{J}_4(\lambda) - \mathcal{J}_5(\lambda)\right.$$

$$+\left[-\int_a^\infty \mathrm{d}\rho\ f\rho^2 g^{(1)} + \int_a^{a\sqrt{2}} \mathrm{d}\rho\ f\rho^2 g^{(2)} + \int_{a\sqrt{2}}^{a\sqrt{3}} \mathrm{d}\rho\ f\rho^2 g^{(3)}\right]\Bigg\}. \tag{10.6}$$

As for any macroscopic a, the pure Casimir terms may be evaluated with $f \to C_{\mathrm{CP}}/\rho^7$. The integrals are straightforward in principle but exceedingly tedious over region 3, especially because MAPLE left to its own devices chooses some wrong branches for the inverse trigonometric functions. Eventually one finds

$$-a\int_a^\infty \mathrm{d}\rho\ g^{(1)}/\rho^5 = \pi - 3 = 0.1416$$

$$a\int_a^{a\sqrt{2}} \mathrm{d}\rho\ g^{(2)}/\rho^5 = -13\pi\sqrt{2}/20 + 7\sqrt{2}/120 - 3\pi/10 + 19/5 = 0.0521$$

$$a\int_{a\sqrt{2}}^{a\sqrt{3}} \mathrm{d}\rho\ g^{(3)}/\rho^5 = 13\pi\sqrt{2}/20 - 9\sqrt{2}/8 - 19\pi/30 + 2\sqrt{3}/5 = 0.000\,03$$

$$\Delta B \simeq -(n\alpha)^2\left\{4a\mathcal{J}_4(\lambda) - \frac{1}{2}\mathcal{J}_5(\lambda) + \frac{23}{4\pi a}\left[\frac{2}{5} + \frac{\pi}{30} - \frac{8\sqrt{2}}{15} + \frac{\sqrt{3}}{5}\right]\right\} \tag{10.7}$$

$$[\cdots] = 0.0969.$$

For the cube, unlike the sphere, the pure Casimir component of ΔB is attractive. By contrast, for a perfectly reflecting thin-walled cubical box Lukosz (1971, 1973) and Ambjorn and Wolfram (1983) give the pure Casimir energy as $+0.0916/a$, implying a sign change as $(n\alpha)^2$ rises from zero (see also Actor 1994). Lukosz (1973) argues for an additional divergent term $a/3\pi\Lambda_L^2$, where Λ_L is his inverse cutoff frequency, analogous perhaps to our λ; this is reminiscent of $a\mathcal{J}_4(\lambda)$ in (10.8), though here too the signs differ.

11. Right circular cylinder

We require B/L and $\Delta B/L$, where L stands for the infinite total length of the cylinder (and not, as elsewhere, for the length of some sharp edge). The crucial difference from spheres and cubes is that cylinders contain pairs of elements arbitrarily far apart. On dimensional grounds, pure Casimir components of the renormalized energy $\Delta B/L$ must be of order $1/a^2$, where a is the radius. The calculation is laborious, and is relegated to appendix D; the result is decidedly peculiar.

First, to order $(n\alpha)^2$ *the pure Casimir terms vanish*, though only by virtue of a cancellation unheralded before the very end. For the potential $-C_{\mathrm{CP}}/\rho^7$ this has been observed already by Milonni (private communication 1999), and by Milton *et al* (1999).

Second, in contrast to the sphere, there is no purely shape-dependent term.

Third, *the leading term of* $\Delta B/L$ *diverges as* $\Omega\lambda \to 0$ *and yet depends on the radius* a:

$$\Delta B/L = -(n\alpha)^2\frac{\pi^2}{32}\mathcal{J}_5(\lambda)\frac{1}{a} + (\text{terms vanishing as } \lambda\Omega \to 0 \text{ and } a\Omega \to \infty)$$

$$\simeq -(n\alpha)^2\frac{\Omega}{a}\frac{\pi^2}{128}\left\{\log\left(\frac{1}{2\Omega\lambda}\right) - \gamma + \frac{65}{24}\right\}. \tag{11.1}$$

However, the total energy is dominated as always by $Vu + S\sigma$, tending to make the cylinder contract regardless of the negative sign of ΔB.

A pure Casmir term will presumably appear to higher than second order in $(n\alpha)$. For a perfectly reflecting thin cylindrical shell, DeRaad and Milton (1981) give it as $-0.014/a^2$, confirmed by Gosdzinsky and Romeo (1998); see also Lambiase *et al* (1999).

12. Conclusions and comments

• Our principal conclusions regarding the perturbative Casimir energies of connected macroscopic dielectric bodies, summarized once more and very briefly, are as follows.

(1) Dispersion cures no divergences, though it does affect degrees of divergence. It must be built into the theory from the start.

(2) The physically appropriate way to eliminate divergences from continuum models is to allow for the finite minimum separation between atoms.

(3) The nominally divergent components $Vu + S\sigma$ of the energy dominate, and ensure that Casimir effects always tend to induce contraction rather than expansion. Discarding these components would make it impossible to understand the physics.

(4)There are other nominally divergent components besides. Some stem from sharp edges; others, dependent only on the shape and not on the size of the body, stem from sharp vertices and from the curvature of smooth parts of the surface.

(5) *Pure Casimir terms* may be defined as convergent components of the energy, dependent only on $\hbar$, c, the electrostatic polarizability of the material and the linear dimensions of the body. Such terms govern the forces between disjoint bodies at large separations, but their contributions to the binding energies of connected bodies are so small that they are impossible to detect.

• If one tries to compare weak and strong reflectors, one must bear in mind an important dichotomy: pairwise additive potentials dominate at short range where interactions are strong, though the dipole approximation to $f(\rho)$ may need amendment; while at long range, where interactions are weak, the dipole approximation should be reliable, but non-pairwise-additive interactions cease to be negligible. (For instance, the weakness of three-atom compared to two-atom forces is partially offset by the fact that with rising separations the number of triplets rises much faster than the number of pairs.)

Thus, paradoxically, perturbation theory is the natural way to a reasonable estimate of the large (nominally divergent) components of the total binding energies B; whereas accurate expressions for the far smaller pure Casimir components require, in practice, the nonperturbative zero-point sums.

• The impossibility asserted in conclusion 5 stems from the obvious geometrical fact that no connected body of given mass can change its shape or size without relative displacements of adjacent atoms. These entail short-range energy changes proportional to typical atomic excitation energies $\hbar\Omega$ multiplied by the cube or square of the ratio a/λ of macroscopic to atomic lengths; by contrast, pure Casimir terms are proportional to $\hbar c/a$; and no macroscopic binding energy will ever be measured to an accuracy of

$$(\hbar c/a)\,(\lambda/a)^2/\hbar\Omega = c\lambda^2/\Omega a^3 \sim (\text{optical wavelength}) \times \lambda^2/a^3.$$

• Finally we demonstrate how easily attention can be diverted from nominally divergent but physically essential terms by focusing on perfect reflectors exclusively or too soon. Such reflectors exclude the Maxwell fields completely, encouraging one to believe that nothing has been overlooked as long as the fields outside are described correctly. This belief tallies with the fact that the finite parts of the zero-point energies $\sum_i \omega_i/2$ determined by considering only the exterior fields do yield the archetypal Casimir attraction between parallel perfect mirrors.

Nevertheless the belief is false: plausible though it may appear when one contemplates perfect reflection from the outset, it generally fails in the approach to this limit, which is the only physically relevant way to idealize. Unfortunately, perturbation theory is not well suited to investigating the various mechanisms of such failure, and experience suggests that

it is unwise to generalize about them; hence we merely cite two examples, one with and one without dispersion.

(i) It is well known that the zero-point Maxwell fields outside a flat perfect mirror generate no surface energy, because the contributions from $\langle \boldsymbol{E}^2 \rangle$ and $\langle \boldsymbol{B}^2 \rangle$ are equal and opposite. But if the material is modelled as a hydrodynamic plasma with a bulk plasma frequency $\omega_{\rm p}$ (reflecting perfectly in the limit $\omega_{\rm p}^2 \to \infty$), then it does possess a surface energy $\sigma \sim \omega_{\rm p}/\lambda^2$, where $1/\lambda$ is a wavenumber cutoff comparable to the Fermi momentum (see e.g. Barton 1979). Therefore, far from vanishing as $\omega_{\rm p}^2 \to \infty$, the surface energy actually diverges. Even for fixed finite $\omega_{\rm p}$ it would diverge as $\lambda \to 0$, just as we have seen it do for weak reflectors.

While σ is barely affected by relativity and retardation, the interior can remain important also to essentially relativistic phenomena. For instance, (ii), an oscillating halfspace with a nondispersive refractive index ν emits radiation, and the power flowing into the material remains finite as $\nu \to \infty$ (Barton and North 1996); whereas considering perfect reflection (infinite ν) from the outset one would have expected that waves could not penetrate the material at all.

Indeed one might well conjecture that contributions to electromagnetic energies from the interiors of perfect reflectors are the rule rather than the exception.

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Appendix A. The moments $\mathcal{J}_N(\lambda)$

In the CP regime the convergent $\mathcal{J}_N(\rho)$ are easily approximated through the expansion (2.6) of f in inverse powers of $\Omega\rho$. The problem is to deal with the opposite VdW regime $\Omega\rho \ll 1$, and in particular with the $\mathcal{J}_N(\lambda)$. One reasonably convenient method takes $\int \mathrm{d}\rho \ldots$ under the integral $\int \mathrm{d}x \ldots$ in (2.1), and changes from the integration variable ρ to $y = x/\Omega\rho$:

$$\mathcal{J}_N(\rho) = \frac{\Omega^{6-N}}{\pi} \int_0^\infty \mathrm{d}x \, \exp(-2x) h(x) x^{N-6} \int_0^{x/\Omega\rho} \mathrm{d}y \, \frac{y^{-N+5}}{(y^2+1)^2}.$$

Given $\Omega\rho \ll 1$ one can split $\int \mathrm{d}x$ at say $x = s = \sqrt{\Omega\rho}$. For $x > s$ one expands the inner integral appropriately to $x/\Omega\rho \gg 1$; for $x < s$ one expands $\exp(-2x)h(x)x^{N-6}$ in ascending powers of x, appropriately to $x \ll 1$. All the resulting integrals are elementary, and when they are combined the dummy parameter s naturally cancels, to yield (2.10)–(2.14).

Alternatively, one can find recursion relations that express the functions

$$\mathcal{L}_N(M) \equiv -\frac{\pi(2\Omega)^{N+1}}{32\Omega^7} \mathcal{J}_N(\rho), \qquad M \equiv 2\Omega\rho$$

as integrals over $\mathrm{Ci}(\mu)$ and $\mathrm{si}(\mu)$, and are easy to run through MAPLE, though forbiddingly tedious by hand. The $\mathcal{L}_N$ with even N evolve from the germ

$$\beta_{-1}(M) \equiv -\int_M^\infty \frac{\mathrm{d}\mu}{\mu} \left[\cos\mu \mathrm{Ci}(\mu) + \sin\mu \mathrm{si}(\mu)\right] = \frac{1}{2}\left[\mathrm{Ci}(M)^2 + \mathrm{si}(M)^2\right]$$

and emerge as

$$\mathcal{L}_4(M) = \frac{M^2}{16} - \frac{13}{8} + \left[\sin M \mathrm{Ci}(M) - \cos M \mathrm{si}(M)\right]\left[-\frac{M^3}{16} + \frac{5M}{4} - \frac{3}{M}\right]$$
$$+ \left[\sin M \mathrm{si}(M) + \cos M \mathrm{Ci}(M)\right]\left[-\frac{M^2}{4} + 3\right]$$

$$\mathcal{L}_2(M) = \frac{1}{16} - \frac{1}{M^2} + [\sin M\mathrm{Ci}(M) - \cos M\mathrm{si}(M)]\left[-\frac{M}{16} + \frac{3}{4M} - \frac{1}{M^3}\right] + [\sin M\mathrm{si}(M) + \cos M\mathrm{Ci}(M)]\left[-\frac{1}{8} + \frac{1}{M^2}\right].$$

These relations are exact for all positive M.

Unfortunately the $\mathcal{L}_N$ with odd N evolve from the germ

$$\alpha_{-1}(M) \equiv \int_M^\infty \frac{\mathrm{d}\mu}{\mu}\,[\sin\mu\mathrm{Ci}(\mu) - \cos\mu\mathrm{si}(\mu)]$$

which the writer cannot express in closed form. However, it is not too difficult to show that

$$\alpha_{-1}(M) = \mathrm{si}(M)\mathrm{Ci}(M) - H(M)$$

$$H(M) \equiv 2\int_M^\infty \mathrm{d}\mu\,\frac{\sin\mu}{\mu}\mathrm{Ci}(\mu) = -2\int_0^M \mathrm{d}\mu\,\frac{\sin\mu}{\mu}\mathrm{Ci}(\mu).$$

This makes the recursion relations operable for $M \ll 1$, where $-H(M)$ is a small correction given by its $\int_0^M \mathrm{d}\mu\ldots$ version, with the integrand approximated appropriately to $\mu \ll 1$. The moments found by the split-range method have all been checked by such approximations for odd N, together with the closed-form expressions for even N.

Appendix B. Failure of the cavity theorem for non-additive interactions

Interactions that are not pairwise additive invalidate the theorem. Many-body potentials and the three-atom potential in particular are discussed for instance by Power and Thirunamachandran (1985, 1994).

The disproof runs largely parallel to the proof in section 3.2; we give it with minimal comment, using subscripts 3 to identify contributions from a three-body potential $\mathrm{d}^3r\,\mathrm{d}^3r'\,\mathrm{d}^3r''\,U(\boldsymbol{r},\boldsymbol{r}',\boldsymbol{r}'')$, which affords the simplest counterexample. The argument turns on the elementary combinatorics ensuring that every triplet of volume elements is counted only once, and that in a triplet with one element fixed the other pair of elements is likewise counted only once. Again the potential need only be translation invariant and symmetric in its three arguments.

Evidently

$$B_3^* = \frac{1}{3!}\iiint_{\text{(all space)}} \mathrm{d}^3r\,\mathrm{d}^3r'\,\mathrm{d}^3r''\,U(\boldsymbol{r},\boldsymbol{r}',\boldsymbol{r}'')$$
$$= \tfrac{1}{6}(\mathrm{in},\mathrm{in},\mathrm{in}) + \tfrac{1}{6}(\mathrm{out},\mathrm{out},\mathrm{out}) + \tfrac{1}{2}(\mathrm{in},\mathrm{in},\mathrm{out}) + \tfrac{1}{2}(\mathrm{in},\mathrm{out},\mathrm{out}). \tag{B.1}$$

But now, underscoring to indicate that $\boldsymbol{r}$ is temporarily fixed in (in), we observe that

$$u_3(\boldsymbol{r}) = u_3 = \tfrac{1}{6}(\underline{\mathrm{in}},\mathrm{in},\mathrm{in}) + \tfrac{1}{3}(\underline{\mathrm{in}},\mathrm{in},\mathrm{out}) + \tfrac{1}{6}(\underline{\mathrm{in}},\mathrm{out},\mathrm{out}). \tag{B.2}$$

Then if we do integrate over (in), we find

$$Vu_3 = \tfrac{1}{6}(\mathrm{in},\mathrm{in},\mathrm{in}) + \tfrac{1}{3}(\mathrm{in},\mathrm{in},\mathrm{out}) + \tfrac{1}{6}(\mathrm{in},\mathrm{out},\mathrm{out}). \tag{B.3}$$

The energies B_{3b} and B_{3c} are given, and ΔB_{3b}, ΔB_{3c} are defined, by

$$Vu_3 + S\sigma_3 + \Delta B_{3b} \equiv B_{3b} = \tfrac{1}{6}(\mathrm{in},\mathrm{in},\mathrm{in}) \tag{B.4}$$

$$B_{3c} = \tfrac{1}{6}(\mathrm{out},\mathrm{out},\mathrm{out}),$$

$$-Vu_3 + S\sigma_3 + \Delta B_{3c} \equiv B_{3c} - B_3^* = -\tfrac{1}{6}(\mathrm{in},\mathrm{in},\mathrm{in}) - \tfrac{1}{2}(\mathrm{in},\mathrm{out},\mathrm{out}) - \tfrac{1}{2}(\mathrm{in},\mathrm{in},\mathrm{out}). \tag{B.5}$$

Finally, on eliminating (in, in, in) and (out, out, out) from (B.3)–(B.5) one obtains

$$\Delta B_{3b} - \Delta B_{3c} = \tfrac{1}{6}\{(\mathrm{in},\mathrm{out},\mathrm{out}) - (\mathrm{in},\mathrm{in},\mathrm{out})\} \neq 0. \tag{B.6}$$

□

Appendix C. The Taylor series for $g(\rho)$

C.1. Volume and surface area

Recall the definitions (5.2) and (5.3) of $\tilde{g}(\boldsymbol{\rho})$ and $g(\rho)$, and consider $g(\rho \to 0)$, with $g_n \equiv d^n g/d\rho^n \mid_{\rho=0}$. Introduce the effectively one-dimensional delta function $\delta_S(\boldsymbol{r})$ confining $\boldsymbol{r}$ to the surface S of the body, and define $\boldsymbol{n}$ as the unit inward normal at $\boldsymbol{r}$, so that

$$\int \mathrm{d}^3 r\, \chi(\boldsymbol{r})\delta_S(\boldsymbol{r}) \ldots = \int_S \mathrm{d}S \ldots \tag{C.1}$$

$$\boldsymbol{\nabla}\chi(\boldsymbol{r}) = \boldsymbol{n}\delta_S(\boldsymbol{r}). \tag{C.2}$$

Obviously

$$\tilde{g}(0) = \int \mathrm{d}^3 r\, \chi^2(\boldsymbol{r}) = \int \mathrm{d}^3 r\, \chi(\boldsymbol{r}) = V, \qquad g(0) = 4\pi V. \tag{C.3}$$

To determine g_1, start from

$$\frac{\partial}{\partial\rho}\tilde{g}(\boldsymbol{\rho}) = \iint \mathrm{d}^3 r\, \mathrm{d}^3 r'\, \chi(\boldsymbol{r})\chi(\boldsymbol{r}')\hat{\boldsymbol{\rho}} \cdot \boldsymbol{\nabla}_\rho \delta(\boldsymbol{r}' - \boldsymbol{r} - \boldsymbol{\rho})$$

(hats specify unit vectors). Use $\boldsymbol{\nabla}_\rho \delta(\boldsymbol{r}' - \boldsymbol{r} - \boldsymbol{\rho}) = \boldsymbol{\nabla}_{\boldsymbol{r}} \delta(\boldsymbol{r}' - \boldsymbol{r} - \boldsymbol{\rho})$, and integrate by parts:

$$\begin{aligned}\frac{\partial}{\partial\rho}\tilde{g}(\boldsymbol{\rho}) &= -\int \mathrm{d}^3 r\, \left[\hat{\boldsymbol{\rho}} \cdot \boldsymbol{\nabla}\chi(\boldsymbol{r})\right] \int \mathrm{d}^3 r'\, \chi(\boldsymbol{r}')\delta(\boldsymbol{r}' - \boldsymbol{r} - \boldsymbol{\rho}) \\ &= -\int_S \mathrm{d}S\, \left(\hat{\boldsymbol{\rho}} \cdot \boldsymbol{n}\right) \chi(\boldsymbol{r} + \rho\hat{\boldsymbol{\rho}}) \end{aligned} \tag{C.4}$$

$$\Rightarrow \frac{\delta}{\delta S}\left[\frac{\partial g}{\partial\rho}\right] = -\int \mathrm{d}\Omega_\rho \left(\hat{\boldsymbol{\rho}} \cdot \boldsymbol{n}\right) \chi(\boldsymbol{r} + \rho\hat{\boldsymbol{\rho}}) \equiv -\int \mathrm{d}\Omega_\rho \left(\hat{\boldsymbol{\rho}} \cdot \boldsymbol{n}\right) \chi(\rho\hat{\boldsymbol{\rho}}). \tag{C.5}$$

For convenience, the last step has moved the origin to the point $\boldsymbol{r}$ on S.

As $\rho \to 0$ one can replace the surface by its tangent plane, taken as the xy plane, with the polar axis along $\boldsymbol{n}$. Then $\int \mathrm{d}\Omega$ runs over the inward hemisphere, and $\hat{\boldsymbol{\rho}} \cdot \boldsymbol{n} = \cos\theta$:

$$\frac{\delta g_1}{\delta S} = -2\pi \int_0^1 \mathrm{d}\cos\,\theta \cos\theta = -\pi \Rightarrow g_1 = -\pi S. \tag{C.6}$$

C.2. Smooth surface elements

Consider (C.5) at a point $\boldsymbol{r}$ of a smooth surface element δS. In the tangent plane choose axes that diagonalize the quadratic that locally approximates the equation of S. Write this equation as

$$z = h(x, y) = \tfrac{1}{2}\left(x^2/R_1 + y^2/R_2\right) \tag{C.7}$$

where $R_{1,2}$ are the principal radii of curvature. Then

$$\begin{aligned}\chi(\rho\hat{\boldsymbol{\rho}}) = \Theta\left\{\rho_3 - h(\rho_1, \rho_2)\right\} = \Theta\Bigg\{&\rho\cos\theta - \frac{1}{2R_1}\rho^2 \sin^2\theta\cos^2\phi \\ &-\frac{1}{2R_2}\rho^2\sin^2\theta\sin^2\phi + \cdots\Bigg\}. \end{aligned} \tag{C.8}$$

The argument of Θ is a quadratic in $\cos\theta$, whose roots show that Θ imposes the restriction

$$1 \geqslant \cos\theta \geqslant \frac{\rho}{2}\left(\frac{1}{R_1}\cos^2\phi + \frac{1}{R_2}\sin^2\phi\right) \equiv \mu,$$

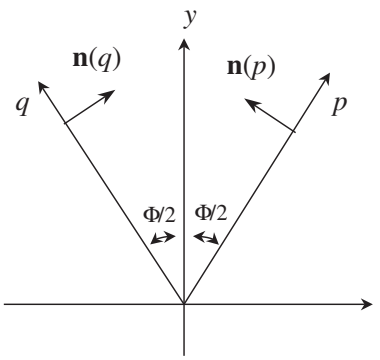

Figure C.1. The z axis is perpendicular to the plane of the paper. The inward unit normals $\boldsymbol{n}(q)$, $\boldsymbol{n}(p)$ are drawn at points whose coordinates are $[x(q), y(q), z']$ and $[x(p), y(p), z]$ respectively.

whence

$$\frac{\delta}{\delta S}\left[\frac{\partial g}{\partial \rho}\right] = -\int_0^{2\pi} \mathrm{d}\phi \int_\mu^1 \mathrm{d}\cos\,\theta\cos\theta$$
$$= -\int_0^{2\pi} \mathrm{d}\phi\,\frac{1}{2}\left\{1-\mu^2+\cdots\right\} = -\pi\left\{1-\frac{\rho^2}{32}\left(\frac{3}{R_1^2}+\frac{2}{R_1R_2}+\frac{3}{R_2^2}\right)+\cdots\right\}.$$

Finally we differentiate twice with respect to ρ, take the limit $\rho \to 0$ and express the result as[9]

$$\frac{\delta g_1}{\delta S} = -\pi, \qquad \frac{\delta g_2}{\delta S} = 0, \qquad \frac{\delta g_3}{\delta S} = \frac{\pi}{16}\left(\frac{3}{R_1^2}+\frac{2}{R_1R_2}+\frac{3}{R_2^2}\right) \quad \text{(for smooth } \delta S\text{)}. \tag{C.9}$$

It must be stressed that sharp edges do induce $g_2 \neq 0$; while the calculation for the cube in section 10 shows that sharp vertices can add to g_3 terms beyond those dictated by (C.9).

C.3. Edges

To determine g_2 we start by differentiating χ as in (C.4), but twice running, which yields

$$\frac{\partial^2 \tilde{g}}{\partial \rho^2} = -\int_S \mathrm{d}S \int_S \mathrm{d}S' \left[\hat{\boldsymbol{\rho}}\cdot\boldsymbol{n}(\boldsymbol{r})\right]\left[\hat{\boldsymbol{\rho}}\cdot\boldsymbol{n}(\boldsymbol{r}')\right]\delta\left[\boldsymbol{r}'-\boldsymbol{r}-\boldsymbol{\rho}\right]. \tag{C.10}$$

Although in a somewhat roundabout way the delta function will ultimately reduce $\int_S \mathrm{d}S \int_S \mathrm{d}S' \ldots$ to $\int_S \mathrm{d}S \ldots$, we cannot write down the result just yet. But we do know that in the limit $\rho \to 0$ an element δS can contribute only if it contains a sharp ridge. Accordingly we envisage a locally straight and flat-sided wedge of opening angle Φ, shown in cross-section in figure C.1, with the z axis perpendicular to the plane of the paper. Thus we have $0 < \Phi < \pi$ if the wedge is convex (as shown), and $\pi < \Phi < 2\pi$ if it is concave. To switch between them one reverses the signs of both $\boldsymbol{n}(\boldsymbol{r})$ and $\boldsymbol{n}(\boldsymbol{r}')$. Since this leaves (C.10) unchanged, $\delta g_2/\delta L$ is the same for a concave as for a convex wedge, i.e. it remains unchanged under $\Phi \to 2\pi - \Phi$, compatibly with the cavity theorem. Therefore it suffices to consider only $0 < \Phi < \pi$.

We require the contribution to (C.10) from an element δS containing an edge of length δL:

$$\int_{\delta S} \mathrm{d}S \cdots = \int_{\delta L} \mathrm{d}z \left\{\int \mathrm{d}q + \int \mathrm{d}p\right\} \ldots \tag{C.11}$$

[9] The ratio 3 : 2 between the numerators in g_3 recurs in the normal-mode density determined by Balian and Duplantier (1978). Deeper relations underlying the coincidence, if there are any, remain unexplored. For general background on mode densities see Baltes and Hilf (1976).

where q is measured perpendicularly to the edge along one flat, p along the other flat, and similarly for $\int_{\delta S} \mathrm{d}S'$. Crucially, the double integration in (C.10) must straddle the edge, because otherwise it merely reproduces the zero for locally smooth (edge-free) elements: in other words contributions to (C.10) survive only if $\int \mathrm{d}S$ features $\int \mathrm{d}q$ while $\int \mathrm{d}S'$ features $\int \mathrm{d}p$, or vice versa. One reaches the same conclusion on observing that if $\boldsymbol{r}$ and $\boldsymbol{r}'$ are both situated on the same face (say on the face coordinatized by q), then the vector $\boldsymbol{\rho} = \boldsymbol{r} - \boldsymbol{r}'$ too lies on that face, so that $\hat{\boldsymbol{\rho}} \cdot \boldsymbol{n}(\boldsymbol{r}) = 0 = \hat{\boldsymbol{\rho}} \cdot \boldsymbol{n}(\boldsymbol{r}')$. Accordingly

$$\delta\left[\boldsymbol{r}' - \boldsymbol{r} - \boldsymbol{\rho}\right] \to \delta\left[z' - z - \rho_3\right] \delta\left[x'(q) - x(p) - \rho_1\right] \delta\left[y'(q) - y(p) - \rho_2\right]. \tag{C.12}$$

Substitution from (C.11) and (C.12) into (C.10) yields

$$\frac{\delta}{\delta L}\frac{\partial^2 \tilde{g}}{\partial \rho^2} = -2\int_0^\infty \int_0^\infty \mathrm{d}q\, \mathrm{d}p \left(\hat{\boldsymbol{\rho}} \cdot \boldsymbol{n}(q)\right)\left(\hat{\boldsymbol{\rho}} \cdot \boldsymbol{n}(p)\right) \delta\left[x'(q) - x(p) - \rho_1\right] \times \delta\left[y'(q) - y(p) - \rho_2\right]. \tag{C.13}$$

The remaining integrations are tedious. Choose the edge, i.e. the z axis, as the polar axis, and let (ρ, θ, ϕ) be the spherical-polar components of $\boldsymbol{\rho}$. One comparatively convenient procedure scales $(q,\ p) \equiv (\eta\rho,\ \eta'\rho)$, and changes variables to $X = \eta' + \eta, \xi = \eta' - \eta$. In effect ρ has already cancelled from the right, so that we can replace $\partial^2 \tilde{g}/\partial\rho^2 \to \tilde{g}_2$. Eventually these steps lead to

$$\begin{aligned}\frac{\delta \tilde{g}_2}{\delta L} &= \frac{\sin^2\theta\,[\cos(2\phi) + \cos(\Phi)]}{\sin(\Phi)} \int_0^\infty \mathrm{d}X \int_{-X}^{X} \mathrm{d}\xi\, \delta\left[X - \frac{\sin\theta\cos\phi}{\cos(\Phi/2)}\right] \delta\left[\xi - \frac{\sin\theta\sin\phi}{\cos(\Phi/2)}\right] \\ &= \frac{\sin^2\theta\,[\cos(2\phi) + \cos(\Phi)]}{\sin(\Phi)} \Theta\left[\frac{\pi}{2} - \frac{\Phi}{2} - |\phi|\right]\end{aligned} \tag{C.14}$$

which in turn yields the end-result

$$\begin{aligned}&\frac{\delta g_2}{\delta L} = \int \mathrm{d}\Omega\, \frac{\delta \tilde{g}_2}{\delta L} = 2\int_0^\pi \mathrm{d}\theta\, \sin\theta \int_{-(\pi-\Phi)/2}^{(\pi-\Phi)/2} \mathrm{d}\phi\, \frac{\sin^2\theta}{\sin(\Phi)} \frac{1}{2}\left[\cos(2\phi) + \cos(\Phi)\right], \\ &\frac{\delta g_2(\Phi)}{\delta L} = \frac{4}{3} Z(\Phi), \qquad Z(\Phi) \equiv 1 + \frac{\pi - \Phi}{\tan(\Phi)}.\end{aligned} \tag{C.15}$$

One checks that this is indeed invariant under $\Phi \to 2\pi - \Phi$. Observe that

$$\frac{\delta g_2}{\delta L}(\Phi \to 0) \to \frac{4\pi}{3\Phi} \to \infty, \qquad \frac{\delta g_2}{\delta L}(\Phi = \pi/2) = \frac{4}{3}$$

while, appropriately to a smooth surface,

$$\frac{\delta g_2}{\delta L}(\Phi = \pi - \varepsilon \to \pi) \to \frac{4}{3}\left\{1 + \frac{\varepsilon}{(-\varepsilon)}\right\} = 0.$$

Appendix D. The cylinder

D.1. Correlation function

Consider a right circular cylinder of radius a and infinite length L, centred on the z axis. We require g/L.

Define $\boldsymbol{\rho} \equiv (\boldsymbol{\sigma}, \zeta)$, $\boldsymbol{k} \equiv (\boldsymbol{\kappa}, p)$ and $\boldsymbol{r} \equiv (\boldsymbol{s}, z)$, where $\boldsymbol{\sigma}$, $\boldsymbol{\kappa}$, $\boldsymbol{s}$ are two-component vectors. The characteristic function and the form-factor read

$$\chi(\boldsymbol{r}) = \Theta(a - s)$$

$$F(\boldsymbol{k}) = \int_{-\infty}^{\infty} \mathrm{d}z\, \exp(\mathrm{i}pz) \int_0^a \mathrm{d}s\, s \int_0^{2\pi} \mathrm{d}\phi \exp(\mathrm{i}\kappa s\cos\phi) = 2\pi\delta(p)\frac{2\pi a}{\kappa} J_1(a\kappa).$$

We identify $[2\pi\delta(p)]^2 = 2\pi\delta(p)L$, and find

$$\frac{\tilde{g}(\boldsymbol{\rho})}{L} = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \mathrm{d}p\ \exp(-\mathrm{i}p\zeta)2\pi\delta(p)\int \mathrm{d}^2\kappa\ \exp(-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\sigma})\left(\frac{2\pi a}{\kappa}\right)^2 J_1^2(a\kappa)$$
$$= 2\pi a^2\int_0^{\infty}\frac{\mathrm{d}\kappa}{\kappa}J_1^2(a\kappa)J_0(\sigma\kappa).$$

Integrating over the polar angles (θ, ϕ) of $\boldsymbol{\rho}$, with $\sigma = \rho\sin\theta$, one obtains

$$\frac{g(\rho)}{L} = 2\pi a^2\int_0^{\infty}\frac{\mathrm{d}\kappa}{\kappa}J_1^2(a\kappa)\int \mathrm{d}\Omega_{\boldsymbol{\rho}}\,J_0(\rho\sigma) = (2\pi a)^2\int_0^{\infty}\frac{\mathrm{d}\kappa}{\kappa}J_1^2(a\kappa)$$
$$\times\int_0^{\pi}\mathrm{d}\theta\ \sin\theta J_0(\rho\kappa\sin\theta).$$

The rightmost integral is standard (Abramowitz and Stegun 1965, equation (11.4.10)), and leaves us with

$$\frac{g(\rho)}{L} = (2\pi a)^2\frac{2}{\rho}\int_0^{\infty}\frac{\mathrm{d}\kappa}{\kappa^2}J_1^2(a\kappa)\sin(\kappa\rho). \tag{D.1}$$

Choosing dimensionless variables

$$z \equiv \rho/2a, \qquad x \equiv \kappa a, \qquad m \equiv \lambda/2a \tag{D.2}$$

we substitute (D.1) into (5.1) to find

$$B/L = -(n\alpha)^2 16\pi^2 a^5\int_m^{\infty}\mathrm{d}z\ f(2az)z\mathcal{G}(z) \tag{D.3}$$

where

$$\mathcal{G}(z) \equiv \frac{\rho g/L}{8\pi^2a^3} = \frac{zg/L}{(2\pi a)^2} = \int_0^{\infty}\frac{\mathrm{d}x}{x^2}\sin(2zx)J_1^2(x) = \sqrt{\pi z}\int_0^{\infty}\frac{\mathrm{d}x}{x^{3/2}}J_{1/2}(2zx)J_1^2(x). \tag{D.4}$$

Some manipulation of the generalized Weber–Schafheitlin integral on page 411 of Watson (1944) yields

$$\mathcal{G}(z) = \frac{1}{2\pi}\int_0^{\pi}\mathrm{d}\phi\,\frac{\sin^2\phi}{\sin(\phi/2)}\int_0^{\infty}\frac{\mathrm{d}t}{t}\sin(2zt)J_1(\omega t), \qquad \omega \equiv 2\sin(\phi/2)$$

while Gröbner and Hofreiter (1958) give

$$\int_0^{\infty}\frac{\mathrm{d}t}{t}\sin(2zt)J_1(\omega t) = \begin{cases} 2z/\omega & \text{for} \quad z < \omega/2 \\ \omega/\left[2z + \sqrt{4z^2-\omega^2}\right] & \text{for} \quad z > \omega/2. \end{cases}$$

On changing to the variable $x = \sin^{-1}[(1/z)\sin(\phi/2)]$ the remaining integrations become manageable though wearisome, and eventually one finds

$$\mathcal{G}(z<1) = z - z^2 + z^2\mathcal{K}(z), \qquad \mathcal{G}(z>1) = z\mathcal{K}(1/z) \tag{D.5}$$

$$\mathcal{K}(z<1) \equiv \frac{4}{\pi}\int_0^1 \mathrm{d}y\,\sqrt{1-y^2}\left[1-\sqrt{1-y^2z^2}\right]$$
$$= 1 - \frac{4}{\pi}\int_0^1\mathrm{d}y\,\sqrt{1-y^2}\sqrt{1-y^2z^2}. \tag{D.6}$$

Then $\mathcal{K}(z)$ expands as

$$\mathcal{K}(z<1) = \frac{z^2}{8} + \frac{z^4}{64} + \frac{5z^6}{1024} + \cdots \tag{D.7}$$

while $\mathcal{K}(1/z)$ expands similarly in powers of $1/z^2$.

There ought to be shorter ways to (D.5) and (D.6), but the writer has failed to find one.

D.2. Energy

For the unrenormalized energy equations (D.3) and (D.5) yield

$$\frac{B}{L} = -(n\alpha)^2 16\pi^2 a^5 \left\{ \int_m^1 \mathrm{d}z\, f(2az) \left[z^2 - z^3 + z^3 \mathcal{K}(z) \right] + \int_1^\infty \mathrm{d}z\, f(2az) z^2 \mathcal{K}\left(\frac{1}{z}\right) \right\}. \tag{D.8}$$

As expected, the first two terms of the first integrand, if the integral were extended to infinity, would contribute $\pi a^2 u + 2\pi a\sigma = (Vu + S\sigma)/L$. To obtain the renormalized energy one adds and subtracts the corresponding integral $\int_1^\infty \mathrm{d}z \ldots$:

$$\frac{\Delta B}{L} = -(n\alpha)^2 16\pi^2 a^5 \left\{ \int_m^1 \mathrm{d}z\, f(2az) z^3 \mathcal{K}(z) + \int_1^\infty \mathrm{d}z\, f(2az) \left[-z^2 + z^3 + z^2 \mathcal{K}\left(\frac{1}{z}\right) \right] \right\}. \tag{D.9}$$

By hindsight, we subdivide

$$\Delta B = \Delta B_{\mathrm{div}} + \Delta B_{\mathrm{conv}}$$

into a divergent part ΔB_{div} proportional to $\mathcal{J}_5(\lambda)$, plus a convergent part ΔB_{conv}. To this end we first split

$$\mathcal{K}(z) = z^2/8 + \mathcal{L}(z), \qquad \mathcal{L}(z \to 0) \sim z^4$$

and then rearrange (D.9) into

$$\Delta B/L = -(n\alpha)^2 16\pi^2 a^5 \left\{ \tfrac{1}{8} \int_m^\infty \mathrm{d}z\, f(2az) z^5 + \int_m^1 \mathrm{d}z\, f(2az) z^3 \mathcal{L}(z) + \int_1^\infty \mathrm{d}z\, f(2az) \left[-z^2 + z^3 + z^2 \mathcal{K}(1/z) - \tfrac{1}{8} z^5 \right] \right\}. \tag{D.10}$$

The first integral inside the curly brackets yields

$$\Delta B_{\mathrm{div}}/L = -(n\alpha)^2 16\pi^2 a^5 \frac{1}{8} \int_m^\infty \mathrm{d}z\, f(2az) z^5 = -(n\alpha)^2 \frac{\pi^2}{32a} \mathcal{J}_5(\lambda)$$
$$\simeq -(n\alpha)^2 \frac{3\pi^2 \Omega}{128a} \left\{ \log\left(\frac{1}{2\Omega\lambda}\right) - \gamma + \frac{65}{24} + \mathcal{O}\left[\Omega\lambda \log\left(\frac{1}{2\Omega\lambda}\right) \right] \right\}. \tag{D.11}$$

Crucially, to leading order in $\lambda/a \ll 1$ we may approximate the middle integral in (D.10) by replacing $\int_m^1 \mathrm{d}z \ldots \to \int_0^1 \mathrm{d}z \ldots$, since $f z^3 \mathcal{L}$ remains finite as $z \to 0$; and then, accurately to leading order in $\Omega a \gg 1$, we may further replace[10] $f \to f_{\mathrm{CP}}$. The same replacement is also made (as always) under $\int_1^\infty \mathrm{d}z \ldots$, where we then change the integration variable from z to $1/z$, so that this integral too runs from 0 to 1. These steps lead to

$$\Delta B_{\mathrm{conv}}/L \simeq -(n\alpha)^2 \frac{23\pi}{32a^2} \int_0^1 \mathrm{d}z \left[\frac{1}{z^4} \mathcal{L}(z) - \frac{1}{8} + z^2 - z^3 + z^3 \mathcal{K}(z) \right]$$
$$= -(n\alpha)^2 \frac{23\pi}{32a^2} \left\{ -\frac{1}{24} + \int_0^1 \mathrm{d}z \left[\frac{1}{z^4} \mathcal{L}(z) + z^3 \mathcal{K}(z) \right] \right\} = 0 \tag{D.12}$$

showing that

$$\Delta B/L = \Delta B_{\mathrm{div}}/L + \frac{(n\alpha)^2}{a^2} \times \{\text{terms that vanish with } \Omega\lambda \text{ and/or } 1/\Omega a\},$$

[10] While this sequence of approximations can prove that $\Delta B/L$ lacks terms of order $1/a^2$, it cannot estimate the correction invoked in the middle expression in (11.1). One might be tempted to try and do so by repeating the present calculation with the second term on the right of (2.6); but then $m \to 0$ would induce a divergence. Estimating the correction evidently requires more analysis than it would warrant at this stage.

as claimed in section 11.

No deep reasons are apparent for the zero in (D.12). To verify it one first substitutes from (D.6) and then integrates over z before y to derive

$$\int_0^1 \mathrm{d}z\, z^3\mathcal{K}(z) = \int_0^1 \mathrm{d}z\, \left\{z^3 - \frac{4z^3}{\pi}\int_0^1 \mathrm{d}y\, \sqrt{1-y^2}\sqrt{1-y^2z^2}\right\} = \frac{1}{4} - \frac{32}{45\pi}. \tag{D.13}$$

In

$$N \equiv \int_0^1 \frac{\mathrm{d}z}{z^4}\mathcal{L}(z) = \int_0^1 \frac{\mathrm{d}z}{z^4}\left\{1 - \frac{z^2}{8} - \frac{4}{\pi}\int_0^1 \mathrm{d}y\, \sqrt{1-y^2}\sqrt{1-y^2z^2}\right\}$$

one integrates by parts twice with respect to y; eventually this yields

$$\begin{aligned} N = \int_0^1 \mathrm{d}z\, \Bigg\{ & \frac{1}{z^4}\left[1 - \frac{z^2}{8} - \sqrt{1-z^2} - \frac{3z^2}{8\sqrt{1-z^2}}\right] - \frac{2}{\pi}\int_0^1 \frac{\mathrm{d}y}{\left(1-y^2z^2\right)^{3/2}} \\ & \times\left[-\frac{y^2}{4}(1-y^2)^{3/2} + \frac{3y^2}{8}(1-y^2)^{1/2} - \frac{y}{8}\sin^{-1}(y) + \frac{y^3}{2}\sin^{-1}(y)\right]\Bigg\} \\ = & -\frac{5}{24} + \frac{32}{45\pi} \end{aligned} \tag{D.14}$$

where the single integral is elementary, and the double integral becomes manageable if $\int_0^1 \mathrm{d}z/\left(1-y^2z^2\right)^{3/2} = 1/\sqrt{1-y^2}$ is performed first. Equations (D.13) and (D.14) then confirm (D.12).

Looking back over the calculation, we see that in order to prove the pure Casimir terms absent, the approximation $f \to f_{\mathrm{CP}}$ is sufficient, and perfectly legitimate. The true potential f is needed only to determine $\mathcal{J}_5(\lambda)$, i.e. $\Delta B_{\mathrm{div}}/L$ as given by (D.11); if f_{CP} were used for this purpose too, then one would conclude, wrongly, that $\Delta B/L = \Delta B_{\mathrm{CP}}/L$, where

$$\Delta B_{\mathrm{CP}}/L = -(n\alpha)^2\frac{23\pi}{128a\lambda} \quad \text{(true but inapplicable)}. \tag{D.15}$$

Appendix E. On the quantum electrodynamics of continuum models

Very briefly, we recall from I the Hamiltonian for a nondispersive dielectric, and sketch how our present dispersive model relates to standard quantum electrodynamics. Both layouts are designed to facilitate perturbative calculations to low orders. In the process we review the traditional frequency cutoff $1/\tilde{\lambda}$ implemented in I, which the writer now thinks is misconceived (for the reasons explained in section 1, and near the end of section E.1 below).

Here one must mention the fundamental work of Renne (1971a, b) on discrete assemblies of harmonic oscillators, formulating a granular nearly-real-world quantum theory of dispersive dielectrics. To this theory the dispersive continuum model is but an approximation, appropriate within limits visible from our motivation of the geometric cutoff $\rho > \lambda$.

E.1. Nondispersive dielectric, and some generalities

Given a body with characteristic function $\chi(\boldsymbol{r})$ and frequency-independent dielectric constant ε, we define the dielectric function $\varepsilon(\boldsymbol{r}) = \varepsilon\chi(\boldsymbol{r}) + (1 - \chi(\boldsymbol{r}))$. The Hamiltonian density reads[11]

$$\mathcal{H}(\boldsymbol{r}) = \mathcal{H}_0(\boldsymbol{r}) + \Delta\mathcal{H}(\boldsymbol{r}) \tag{E.1}$$

[11] This section works in the pseudo-Coulomb gauge, where $\mathrm{div}\boldsymbol{A}$ vanishes except on surfaces of discontinuity of $\varepsilon(\boldsymbol{r})$. The next section works with minimal coupling in the ordinary Coulomb gauge. Recall that α is an atomic polarizability, and not the fine-structure constant $e^2/4\pi \simeq 1/137$.

$$\mathcal{H}_0 = (\boldsymbol{D}^2 + \boldsymbol{B}^2)/2, \qquad \Delta\mathcal{H}(\boldsymbol{r}) = -\frac{\gamma}{2}\chi(\boldsymbol{r})\boldsymbol{D}^2(\boldsymbol{r}), \qquad \gamma \equiv 1 - 1/\varepsilon \simeq 4\pi n\alpha. \tag{E.2}$$

To calculate perturbatively in the Schrödinger picture, we expand $\boldsymbol{D}$ exactly as if the body were absent,

$$\boldsymbol{D}(\boldsymbol{r}) = \mathrm{i}\sum_{s=1,2}\int \mathrm{d}^3k\,\frac{k^{1/2}}{4\pi^{3/2}}\epsilon_{\boldsymbol{k}s}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})a_{\boldsymbol{k}s} + (\text{Hermitean conjugate}); \tag{E.3}$$

note

$$H_0 = \int \mathrm{d}^3r\,\mathcal{H}_0 = \sum_{s=1,2}\int \mathrm{d}^3k\,k\left(a^+_{\boldsymbol{k}s}a_{\boldsymbol{k}s} + 1/2\right); \tag{E.4}$$

and treat $H_{\mathrm{int}} = \int \mathrm{d}^3r\,\Delta\mathcal{H}(\boldsymbol{r})$ as the perturbation. In sums over virtual photons, $\int \mathrm{d}^3k\ldots$ would diverge without a cutoff.

The first-order shift, given by a sum over virtual single-photon states, reads

$$\Delta E^{(1)} = \langle H_{\mathrm{int}}\rangle = \int \mathrm{d}^3r\,\langle 0\,|\Delta\mathcal{H}(\boldsymbol{r})|\,0\rangle = -(\gamma/2)V\langle 0\big|\boldsymbol{D}^2\big|0\rangle. \tag{E.5}$$

It is the model's attempt to reproduce the sum of the self-energies (Lamb shifts) of the separate volume elements of the body, recognizable as such because (a) being proportional to α it is of order e^2, and (b) being proportional to n it is additive, i.e. unaffected by any interaction between different elements. In I, $\Delta E^{(1)}$ could not be specified satisfactorily: its connection with Lamb shifts becomes explicit and persuasive only on introducing dispersion, as in the work of Milonni and Lerner (1992), Schaden *et al* (1998) and Milonni *et al* (1999). However, we discard $\Delta E^{(1)}$ in any case, because it is manifestly irrelevant to the binding energies B that we we wish to consider.

Subject to the wavenumber cutoff $\exp(-\tilde{\lambda}k)$ imposed in I, the second-order shift, given by a sum over virtual two-photon states, eventually turns out to be

$$\begin{aligned} B = \Delta E^{(2)} &= -\tfrac{1}{2}(n\alpha)^2\iint \mathrm{d}^3r\,\mathrm{d}^3r'\,\chi(\boldsymbol{r})\chi(\boldsymbol{r}')f(\rho,\tilde{\lambda}) \\ &= -(n\alpha)^2\tfrac{1}{2}\int_0^\infty \mathrm{d}\rho\,f(\rho,\tilde{\lambda})\rho^2 g(\rho) \end{aligned} \tag{E.6}$$

$$f(\rho,\tilde{\lambda}) = \frac{4}{\pi^2\rho^7}\int_{\tilde{\lambda}/\rho}^\infty \mathrm{d}x\,\frac{12x^4 - 8x^2 + 12}{(x^2+1)^6}. \tag{E.7}$$

One verifies that $f(\rho,0) = f_{\mathrm{CP}}(\rho) = 23/4\pi\rho^7$, while $f(0,\tilde{\lambda}) = 48/7\pi^2\tilde{\lambda}^7$. But there is no good reason why the potential at small distances should be represented specifically as $f(\rho,\tilde{\lambda})$: in fact this would be a thoroughly misleading way to mimic the consequences of dispersion, which for small $\rho\Omega$ causes the true potential $f(\rho)$ to modulate, not into $f(\rho,\tilde{\lambda})$, but into the quite different function $f_{\mathrm{VdW}}(\rho) = C_{\mathrm{VdW}}/\rho^6$. The differences are drastic: with f_{VdW} instead of $f(\rho,\tilde{\lambda})$, the integral (E.6) diverges unless we impose some other cutoff quite unrelated to dispersion, like the condition $\rho > \lambda$ motivated in section 2. In other words, consequences of (E.6) and (E.7) featuring $\tilde{\lambda}$ cannot be trusted, not even semi-quantitatively.

Clearly, any model worth its keep must incorporate dispersion, compatibly with causality, and from the start. If one wishes to concoct a nondispersive model nevertheless, then the (very poor) best one can do is to rewrite (E.6) with $f_{\mathrm{CP}}(\rho)$, and restrict the integral to $\rho > \lambda$. For a sphere for instance this yields (7.6), where the pure Casimir term is correct, but where all the other terms diverge more strongly than they should, i.e. more strongly than in (7.3).

However, (E.6) does show that perturbative QED yields identically the same energies that one gets from two-body potentials $f(\rho)$, provided the $f(\rho)$ are chosen consistently with the continuum model for the material, whatever the shortcomings of the latter.

E.2. Dispersive dielectric

Nondispersive models are relatively easy to handle, because the only dynamical degrees of freedom are those of the Maxwell field. By contrast, dispersion depends on additional degrees of freedom carried by the material, and fog descends unless these are introduced explicitly. For the material envisaged in the text one can adopt a hydrodynamic model[12] with mechanical (i.e. not explicitly electromagnetic) harmonic restoring forces. This model features a continuous fluid with mass and charge densities nm and ne; the displacement of the fluid from equilibrium is $\boldsymbol{\xi}(\boldsymbol{r})$, defined only inside the body and treated as small enough for the equations of motion to be linearized $\boldsymbol{\xi}$; and there is an immobile overall-neutralizing background charge density $-ne$, so that the charge and current densities read $-ne\mathrm{div}\boldsymbol{\xi}$ and $ne\dot{\boldsymbol{\xi}}$. The canonical conjugate of $\boldsymbol{\xi}(\boldsymbol{r})$ is written $\boldsymbol{\Pi}(\boldsymbol{r})$: for a neutral fluid it would equal $nm\dot{\boldsymbol{\xi}}(\boldsymbol{r})$. This section omits factors $\chi(\boldsymbol{r})$.

In the Coulomb gauge the Hamiltonian density reads

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{fluid}} + \mathcal{H}_{\text{int}} + \mathcal{H}_{\text{Coulomb}}. \tag{E.8}$$

$\mathcal{H}_0$, given by (E.2), describes the free Maxwell field in empty space;

$$\mathcal{H}_{\text{fluid}} = \frac{1}{2nm}\boldsymbol{\Pi}^2 + \frac{nm\Omega^2}{2}\boldsymbol{\xi}^2 \tag{E.9}$$

$$\mathcal{H}_{\text{int}} = -\frac{e}{2m}\left(\boldsymbol{\Pi}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{\Pi}\right) + \frac{ne^2}{2m}\boldsymbol{A}^2 \tag{E.10}$$

$$\mathcal{H}_{\text{Coulomb}} = -\tfrac{1}{2}\Phi ne\mathrm{div}\boldsymbol{\xi} \qquad \text{where} \quad -\boldsymbol{\nabla}^2\Phi = -ne\mathrm{div}\boldsymbol{\xi}. \tag{E.11}$$

We treat the frequency Ω as already renormalized, and suppress the attendant armoury of counterterms (Barton 2000). The zero-order Hamiltonian is $\int \mathrm{d}^3r\ \{\mathcal{H}_0 + \chi(\boldsymbol{r})\mathcal{H}_{\text{fluid}}\}$.

Here one meets a problem that, fortunately, perturbation theory can sidestep, albeit only by virtue of the restoring forces. Nonperturbatively one would want to quantize through the exact normal modes of the system; to determine them one must first solve the coupled linear equations for $\boldsymbol{A}$ and $\boldsymbol{\xi}$, equations that require boundary conditions on $\boldsymbol{\xi}$ at the surface of the body; whence one must choose between (a) imposing plausible boundary conditions from the start, which makes the sequel somewhat arbitrary, or (b) trying to establish such conditions through prior arguments, which are always excessively tedious, and not wholly convincing either, seeing how crude the underlying model is in the first place.

By contrast, perturbatively we can proceed as follows. The Maxwell field is quantized as if in free space, validating (E.3) in particular. To quantize the matter field we take advantage of the assumption that the mechanical restoring forces are *local*, i.e. that the Hamiltonian contains no gradients of the ξ_i, and impose the standard equal-time commutation rule[13] $[\xi_i(\boldsymbol{r}), \Pi_j(\boldsymbol{r}')] = \mathrm{i}\delta_{ij}\delta(\boldsymbol{r}-\boldsymbol{r}')$. No special provisions are required for points on the boundary. We need merely imagine the body divided into small cells, treat each cell as having its own material degrees of freedom (namely the ξ_i at some point in the cell), proceed with the calculation, and at the end take the limit where the cell volumes shrink to zero and the number of cells goes to infinity. This is just the prelapsarian way to canonical methods for continuous systems (see e.g. Wentzel 1949). Each cell behaves like a localized harmonic oscillator; to order e^2 the perturbation $\int \mathrm{d}^3r\ (\mathcal{H}_{\text{int}} + \mathcal{H}_{\text{Coulomb}})$ delivers their separate Lamb shifts; to order e^4

[12] Nonrelativistic versions of such models have been explored in some detail to describe VdW images reflected by dielectric surfaces (Barton 2000), and (without restoring forces) surface effects in metallic plasmas (Barton 1979).

[13] Here it is crucial that there are no constraints forcing $\boldsymbol{\xi}(\boldsymbol{r})$ to be purely longitudinal or purely transverse. In this one respect it is actually easier to work with the full Maxwell field than with nonrelativistic approximations using only the instantaneous Coulomb potential. The latter make it possible and convenient to write $\boldsymbol{\xi} = -\boldsymbol{\nabla}\boldsymbol{\Psi}$, but have to be paid for by quite complicated commutators for the scalar displacement-potential Ψ (see e.g. Barton 1979, 2000).

it delivers their mutual interactions[14]; and the final limit reproduces precisely the continuum model considered in the text.

Appendix F. Numbers for solid Ar

We aim to show that, even though our model is so crude, it can talk sense at least as regards orders of magnitude. For comparison we choose argon (at $T = 0$), an archetypal molecular solid, free of the complications stemming from the intense zero-point vibrations that plague solid helium. We use atomic units (au): unit energy = 2 Ry = 27.2 eV = 4.35×10^{-11} erg, unit distance = 0.529 Å, and lightspeed $c = 137$ au. The lattice is close packed, with a packing fraction (pf) = $\pi\sqrt{2}/6 = 0.740$; call the nearest-neighbour distance $R_{\rm nn}$, this being twice the radius of notional hard spheres packing the lattice.

Measurements (Beaumont *et al* 1961) yield

$$n^{-1} = 22.6\text{cm}^3/\text{mole} = (6.33\text{ au})^3/\text{atom} \Rightarrow R_{\rm nn} = 2\left[\frac{3}{4\pi}\frac{(\text{pf})}{n}\right]^{1/3} = 7.11\text{ au}$$

Binding energy = 7.73×10^{10} erg/mole = 2.96×10^{-3} au/atom $\equiv E_0$

$\Rightarrow u_{\rm exp} = -E_0/n = -1.16 \times 10^{-5}$ au.

To the optical data at zero frequency (Sonntag 1976a) we apply the Clausius–Mossotti formula:

$$\varepsilon \equiv \varepsilon(\omega = 0) = 1.56 \Rightarrow 4\pi n\alpha = \frac{3(\varepsilon - 1)}{\varepsilon + 2} = 0.472$$

$$n\alpha = 0.0376, \qquad (n\alpha)^2 = 1.41 \times 10^{-3}.$$

Hence it need not be unreasonable to work only to leading order in $n\alpha$. For the electrostatic polarizability of the atom these solid-state data[15] imply $\alpha = 9.55 = (2.12)^3$ au.

For the excitation frequency we adopt $\Omega = 12$ eV $= 0.44$ au, as suggested by measurements on the dielectric function of the solid (Sonntag 1976b); and we choose the cutoff distance λ so as to validate[16,17]

$$u_{\rm exp} = u = -(n\alpha)^2 2\pi \mathcal{J}_2(\lambda) \simeq (n\alpha)^2 \frac{\pi\Omega}{2\lambda^3} \Rightarrow \lambda = 4.38\text{ au}.$$

The dimensionless parameter $\Omega\lambda/c$ (which we have assumed to be small, and have written elsewhere in natural units as $\Omega\lambda$) is given by

$$\Omega\lambda/c = \Omega\lambda/137 = 0.0141.$$

At this point the model predicts a surface energy

$$\sigma = \frac{\mathcal{J}_3(\lambda)}{4\mathcal{J}_2(\lambda)} \mid u \mid \approx \frac{3\lambda}{8} \mid u \mid = 1.91 \times 10^{-5}\text{ au}.$$

[14] The cutoff $\rho > \lambda$ excludes $\mathcal{O}(e^4)$ contributions from the separate Lamb shifts.

[15] The atomic Stark shift yields $\alpha_{\rm atom} = 11.8$. The energies of the strongest atomic dipole excitations are close to 0.43, practically the same as the solid-state value Ω we are just about to quote. For the atomic data see Radzig and Smirnov (1985).

[16] There is no simple way to foresee whether λ thus determined should turn out greater than $R_{\rm nn}$, or less. In favour of $\lambda > R_{\rm nn}$ one might adduce that $\alpha^2 f(R_{\rm nn})$ overestimates the magnitude of the true interatomic potential (since the core is not completely hard): $|u_{\rm exp}|$ would then be fitted with a minimum separation λ larger than the true $R_{\rm nn}$. On the other hand, $\lambda < R_{\rm nn}$ is favoured because perturbation theory calculates the two-body correlation function g from a uniform density: this will underestimate the true gf near $\rho = R_{\rm nn}$, effectively underestimating the attraction, and requiring a smaller minimum separation to reproduce $|u_{\rm exp}|$.

[17] Through (1.2) and (2.1) these values of Ω and λ determine a potential whose implications can be considered directly for a crystal, rather than for our continuum model. Thus one might compare $u_{\rm exp}$ with the contribution from the 12 nearest neighbours, namely with $u_{\rm nn} = -(1/2)12\alpha^2 f(R_{\rm nn})n$. From $\Omega R_{\rm nn}/c = 0.0228 \ll 1$ we see that $f \simeq f_{\rm vdW} = 3\Omega/4R_{\rm nn}^6$ and $u_{\rm nn} = -(9/2)(n\alpha)^2\Omega/nR_{\rm nn}^6 \simeq -0.55 \times 10^{-5}$ au, about half of $u_{\rm exp}$.

With λ replaced by $R_{\rm nn}$ this formula would give 3.1×10^{-5} au. Apparently σ has not been measured, but serious calculations (Adamson 1990) yield $\sigma_{\rm crystal} = 47$ erg cm^{-2} $= 3.0 \times 10^{-5}$ au, depending only weakly on the crystal face.

Finally we digress to quote the calculated three-body contribution u_3 to the binding energy (Bell and Zucker 1976), namely $u_3/u \simeq -0.07$, which might be taken as a measure of deviations to be expected from the cavity theorem.

Note added in proof. As regards comparing *the definitions of renormalization* in conventional field theory and in this paper (cf the italicized comment towards the end of section 1), the writer is grateful to Michael Bordag for stressing another point of view that does reveal a similarity. Each theory introduces certain parameters that are treated essentially as classical, and as given before quantum mechanics is brought to bear: conventionally these are the bare masses and coupling constants, while here they describe the body through $a, \alpha, \Omega, \lambda$, and (if one so chooses) u and σ. Then an obvious analogy appears in that the subdivision $B = Vu + S\sigma + \Delta B$ allows some (though not all) nominal divergences to be absorbed by redefining u and σ. (Of course, there are equally obvious differences: here the divergences are only nominal, and the calculation of B could be construed as determining u and σ *ab inito*, rather than as renormalizing pre-assigned values.)

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